School of Mech & Aero Eng
Seoul National University

Eng Probability
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FINAL

- 'How you arrived at your answer' is much more important than the answer itself. Read the following problems carefully, and make sure you show your work *step by step*.
- You can attach extra pages if necessary. Please use a separate sheet for each problem.
- In case you have forgotten, integration by parts:

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

Density function of a standard normal random variable:

$$f(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

Thank you for your hard work, and **GOOD LUCK !**

Student ID: ______________

Name: ______________

1	/ 10
2	/ 20
3	/ 40
4	/ 15
5	/ 15
EXTRA	$+\alpha$
Total	/ 100

1. [`10 pts=5+5`]

(a) A record claims that the life expectancy of patients with a particular disease has a mean of 54 months and a standard deviation of 3 months. A hospital tests 50 patients with that disease. Assuming the manufacturer's claims are true, what is the probability that the test finds a mean lifetime of longer than 60 months?

(b) Is the following statement true ? Why?

(1) If the two random variables are independent, then they are not correlated.

(2) $E(Yg(X)|X) = E(g(X))E(Y|X)$ for any suitable function g.

2. [`20 pts=6+6+8`] Let X and Y have the joint density

$$f(x,y) = cx(y-x)e^{-y}, \qquad 0 \le x \le y < \infty.$$

(a) Find the value of c.

(b) Show that

$$\begin{aligned} f_{X|Y}(x|y) &= 6x(y-x)y^{-3}, & 0 \le x \le y \\ f_{Y|X}(y|x) &= (y-x)e^{x-y}, & 0 \le x \le y < \infty \end{aligned}$$

(c) Compute $E(X|Y)$ and $E(Y|X)$.

3. [`40 pts=7+7+7+4+5+10`] Let Z_1, Z_2 be independent normal random variables with mean 0 and variance 1.

(a) Let $X_1 = Z_1$, $X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$, where $|\rho| < 1$. Show that

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right]$$

(b) Compute $E(X_2), var(X_2), E(X_1X_2)$, and correlation of X_1 and X_2.

(c) Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint density function f_{Y_1,Y_2} of Y_1 and Y_2.

(d) Are Y_1 and Y_2 independent? Why?

(e) Compute the marginal density functions of Y_1 and Y_2. What kind of distributions do they have ?

(f) Show that

$$P(X_1 > 0, X_2 > 0) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho \, .$$

(`Hint`: it might be easier to represent the above value in terms of Z_1 and Z_2.

4. [`15 pts=7+8`]

(a) Describe the property of the moment generating function of a random variable.

(b) Describe why $\xi(t) = (1+t^4)^{-1}$ cannot be a moment generating function.

5. [15 pts=8+7] Let X, Y, and Z be independent and uniformly distributed on $[0, 1]$.

(a) Compute the joint density function of XY and Z^2.

(b) Show that $P(XY < Z^2) = 5/9$.

EXTRA. Please write down the names of your classmates taking this course (as many as you know).