School of Mech & Aero Eng
Seoul National University

Eng Probability
June 07, 2008

FINAL

- 'How you arrived at your answer' is much more important than the answer itself. Read the following problems carefully, and make sure you show your work *step by step*.
- You can attach extra pages if necessary. Please use a separate sheet for each problem.
- Ask questions if you don't understand what you are being asked.

Thank you for your hard work, and **GOOD LUCK !**

Student ID: ______________

Name: ______________

1	/ 10
2	/ 20
3	/ 40
4	/ 15
5	/ 15
EXTRA	$+\alpha$
Total	/ 100

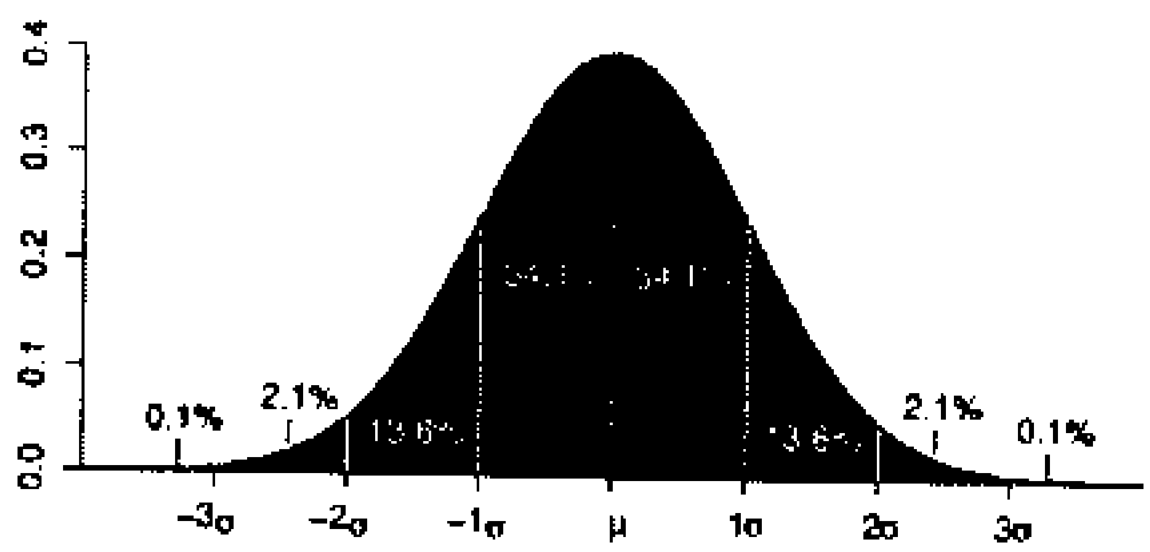
1. [10 pts=5+5]

(a) A record claims that the life expectancy of patients with a particular disease has a mean of 54 months and a standard deviation of 3 months. A hospital tests 50 patients with that disease. Assuming the manufacturer's claims are true, what is the probability that the test finds a mean lifetime of longer than 60 months?

(b) Is the following statement true ? Why?

(1) If the two random variables are independent, then they are not correlated.

(2) $E(Yg(X)|X) = E(g(X))E(Y|X)$ for any suitable function g.



Sol.

(a)

$$P(\bar{X} \geq 60) = P(\frac{\bar{X} - 54}{3} \geq \frac{60 - 54}{3}) \approx 1 - 0.954$$

(b) (1) yes. If X and Y are independent then $cov(X, Y) = E((X - E(X))(Y - E(Y)) = E(X - E(X))E(Y - E(Y)) = 0$.

(2) no. $E(Yg(X)|X) = g(X)E(Y|X)$

$$E(Yg(X)|X=x) = \sum_y y\, g(x)\, P(Y=y, X=x|X=x)$$
$$= g(x) \sum y\, P(Y=y, X=x|X=x)$$
$$= g(x)\, E(Y|X=x)$$
$$\therefore E(Yg(X)|X) = g(X)\, E(Y|X).$$

2. [20 **pts**] Let X and Y have the joint density

$$f(x,y) = cx(y-x)e^{-y}, \quad 0 \le x \le y < \infty.$$

6 (a) Find the value of c.

6 (b) Show that

$$f_{X|Y}(x|y) = 6x(y-x)y^{-3}, \qquad 0 \le x \le y$$
$$f_{Y|X}(y|x) = (y-x)e^{x-y}, \qquad 0 \le x \le y < \infty$$

8 (c) Compute $E(X|Y)$ and $E(Y|X)$.

(a) For $x, y > 0$,

$$f_Y(y) = \int_0^y f(x,y)\,dx = ce^{-y}\int_0^y x(y-x)\,dx = \frac{1}{6}cy^3e^{-y}.$$

$$f_X(x) = \int_x^\infty f(x,y)\,dy = cx\int_x^\infty (y-x)e^{-y}\,dy$$

$$= cx\left[-(y-x)e^{-y}\Big|_x^\infty + \int_x^\infty e^{-y}\,dy\right]$$

$$= cx\left[-(y-x)e^{-y} - e^{-y}\right]_x^\infty$$

$$= cxe^{-x}.$$

$$\int_x^\infty f_X(x)\,dx = 1 \Rightarrow \underline{c = 1.}$$

(b) $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = 6x(y-x)y^{-3}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = (y-x)e^{x-y}$$

(c) $$E(X|Y=y) = \int_0^y 6x^2(y-x)y^{-3}\,dx$$

$$= \left[2x^3y - \frac{3}{2}x^4\right]_0^y \cdot y^{-3}$$

$$= \left(2y^4 - \frac{3}{2}y^4\right)\cdot y^{-3}$$

$$= \frac{y}{2}.$$

$$\therefore \underline{E(X|Y) = \frac{Y}{2}}.$$

$$E(Y|X=x) = \int_x^{\infty} y(y-x)e^{x-y}\,dy$$

$$= e^x \int_x^{\infty} (y^2 - xy)e^{-y}\,dy$$

$$= e^x \left[\cancel{-(y^2-xy)e^{-y}\Big|_x^{\infty}}^{\,0} + \int_x^{\infty} (2y-x)e^{-y}\,dy \right]$$

$$= e^x \left[-(2y-x)e^{-y}\Big|_x^{\infty} + 2\int_x^{\infty} e^{-y}\,dy \right]$$

$$= e^x \left[xe^{-x} + 2e^{-x} \right]$$

$$= (x+2)$$

$$\therefore \ \underline{E(Y|X) = X+2}.$$

3. [`40 pts=7+7+7+4+5+10`] Let Z_1, Z_2 be independent normal random variables with mean 0 and variance 1.

(a) Let $X_1 = Z_1$, $X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$, where $|\rho| < 1$. Show that

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right]$$

(b) Compute $E(X_2), var(X_2), E(X_1X_2)$, and correlation of X_1 and X_2.

(c) Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint density function f_{Y_1,Y_2} of Y_1 and Y_2.

(d) Are Y_1 and Y_2 independent? Why?

(e) Compute the marginal density functions of Y_1 and Y_2. What kind of distributions do they have ?

(f) Show that

$$P(X_1 > 0, X_2 > 0) = \frac{1}{4} + \frac{1}{2\pi} \sin^{\ 1} \rho\,.$$

(`Hint`: it might be easier to represent the above value in terms of Z_1 and Z_2.

Sol.

(a)

$$J(z_1, z_2) = \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{vmatrix} = \sqrt{1-\rho^2}$$

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{\sqrt{2\pi}} \exp{}^{z_1^2/2} \frac{1}{\sqrt{2\pi}} \exp^{-z_2^2/2} = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)}.$$

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2) &= f_{Z_1,Z_2}(z_1(x_1, x_2), z_2(x_1, x_2))|J|^{-1} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(x_1^2 + \frac{(x_2 - \rho x_1)^2}{1-\rho^2}\right)\right] \\ &= answer. \end{aligned}$$

(b)

$$E(X_1) = E(Z_1) = 0,$$

$$E(X_2) = \rho E(Z_1) + \sqrt{1-\rho^2} E(Z_2) = 0$$

$$\begin{aligned}
var(X_1) &= var(Z_1) = 1 \\
var(X_2) = E(X_2^2) &= \rho^2 E(Z_1^2) + (1-\rho^2)E(Z_2^2) = 1 \\
E(X_1X_2) &= E(\rho Z_1^2 + \sqrt{1-\rho^2}Z_1Z_2) = \rho \\
cor(X_1, X_2) &= \frac{E(X_1X_2)}{\sqrt{var(X_1)var(X_2)}} = \rho
\end{aligned}$$

(c)

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J|^{-1} = \frac{1}{2} f_{X_1,X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right)$$

The exponent of the joint density function (called bivariate normal distribution) is

$$-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)} = -\frac{y_1^2(1-\rho) + y_2^2(1+\rho)}{4(1-\rho^2)}$$

Therefore

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left[-\frac{y_1^2}{4(1+\rho)} - \frac{y_2^2}{4(1-\rho)}\right]$$

(d)

$$\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= \frac{1}{\sqrt{2\pi}\,2\sqrt{1+\rho}} \exp\left[-\frac{y_1^2}{4(1+\rho)}\right] \cdot \frac{1}{\sqrt{2\pi}\,2\sqrt{1-\rho}} \exp\left[-\frac{y_2^2}{4(1-\rho)}\right] \\
&= f_{Y_1}(y_1) f_{Y_2}(y_2)\,.
\end{aligned}$$

Thus, Y_1 and Y_2 are independent.

(e)

$$\begin{aligned}
f_{Y_1}(y_1) &= \frac{1}{\sqrt{2\pi}\,2\sqrt{1+\rho}} \exp\left[-\frac{y_1^2}{4(1+\rho)}\right] \\
f_{Y_2}(y_2) &= \frac{1}{\sqrt{2\pi}\,2\sqrt{1-\rho}} \exp\left[-\frac{y_2^2}{4(1-\rho)}\right]
\end{aligned}$$

Thus, $Y_1 \sim \mathcal{N}(0, 2(1+\rho))$ and $Y_2 \sim \mathcal{N}(0, 2(1-\rho))$.

(f)

$$
\begin{aligned}
P(X_1 > 0, X_2 > 0) &= P(Z_1 > 0, \rho Z_1 + \sqrt{1-\rho^2} Z_2 > 0) \\
&= P(Z_1 > 0, Z_2 > -\frac{\rho Z_1}{\sqrt{1-\rho^2}}) \\
&\quad \text{change to polar coordinate} \\
&= \int_\alpha^{\frac{\pi}{2}} \int_0^\infty \frac{1}{2\pi} e^{-r^2/2} r \, dr \, d\theta \\
&= \frac{1}{2\pi} \int_\alpha^{\frac{\pi}{2}} d\theta \\
&= \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho \, .
\end{aligned}
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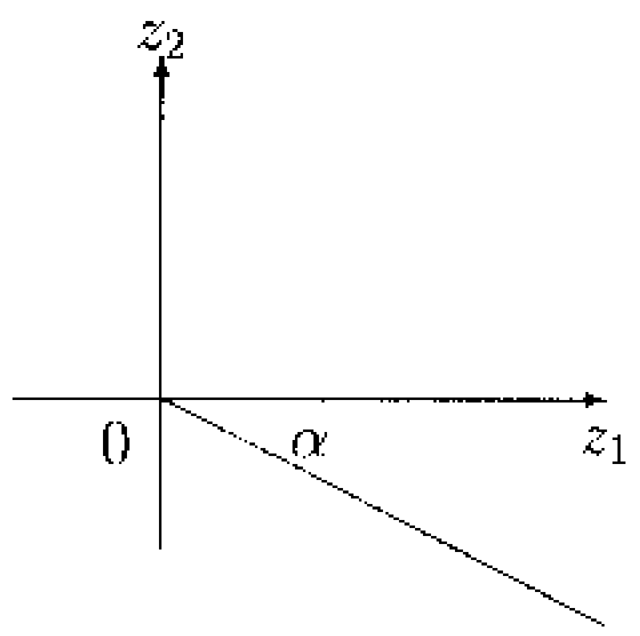


$\tan\alpha = -\frac{\rho}{\sqrt{1-\rho^2}}, \quad \sin\alpha = -\rho$

4. [15 pts]

(a) Describe the property of the moment generating function of a random variable.

(b) Describe why $\xi(t) = (1+t^4)^{-1}$ cannot be a moment generating function.

Sol.

(a)

$$M(t) = E(e^{tX})$$

$$M'(0) = E(X),\ M''(0) = E(X^2), \cdots$$

(b) Suppose ξ is a mo. gen func. Then because $\xi(0) = \xi'(0) = \xi''(0)$, $X = 0$. Thus $M(t) = 1$, which is a contradiction.

5. [15 pts] Let X, Y, and Z be independent and uniformly distributed on $[0,1]$.

(a) Compute the joint density function of XY and Z^2.

(b) Show that $P(XY < Z^2) = 5/9$.

(a) $P(XY \le u,\ Z^2 \le v) \overset{\text{indept}}{=} P(XY \le u)\ P(Z \le \sqrt{v})$

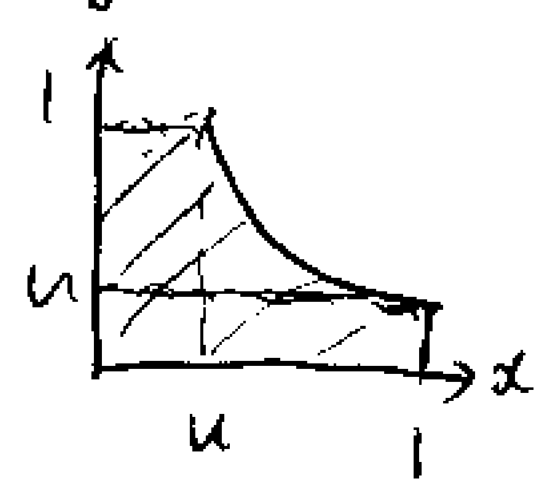
$$P(Z \le \sqrt{v}) = \sqrt{v},\quad 0 \le v \le 1$$

$$P(XY \le u) = P\left(X \le \frac{u}{Y}\right)$$

$$= u + \int_u^1 \int_0^{\frac{u}{y}} dx\, dy$$

$$= u + \int_u^1 \frac{u}{y}\, dy$$

$$= u(1 - \ln u)$$



$$P(XY \le u,\ Z^2 \le v) = \sqrt{v}\, u(1 - \ln u)$$

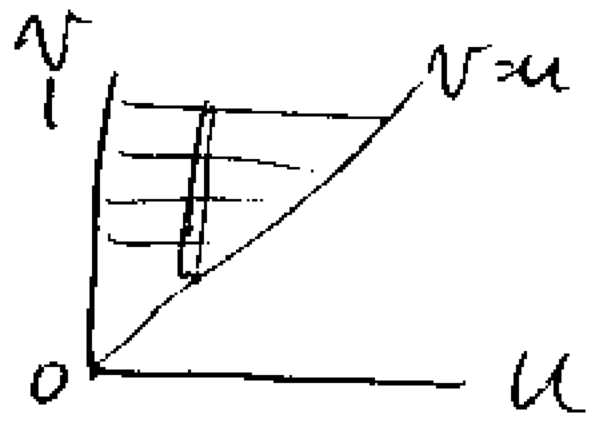
↳ differentiate.

$$g(u,v) = \frac{-\ln u}{2\sqrt{v}} \qquad 0 \le u, v \le 1$$

//

(b) $$P(XY < Z^2) = \iint_{0 \le u \le v \le 1} g(u,v)\, du\, dv$$

$$= \int_0^1 \int_u^1 \frac{-\ln u}{2\sqrt{v}}\, dv\, du$$



$$= \int_0^1 (\sqrt{u}-1)\ln u \, du$$

$$= \cancel{\left(\frac{2}{3}u^{\frac{3}{2}}-u\right)\ln u}\Big|_0^1 - \int_0^1 \left(\frac{2}{3}u^{\frac{1}{2}}-1\right) du$$

$$= -\left[\frac{2}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}-u\right]_0^1$$

$$= 1 - \frac{4}{9} = \frac{5}{9}.$$

EXTRA. Please write down the names of your classmates taking this course.