SIGNAL and SYSTEMS

May 24, 2008

(Instructor : In-Joong Ha, Professor)

EXAM II

[Problem 1] (10 Points) Consider the signal $x(t)$ in Fig. 1 below.

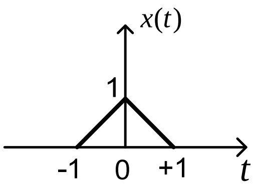
(a) Find the Fourier transform $X(jw)$ of $x(t)$.

(b) Consider the new signal $\tilde{x}(t)$ defined as

$$\tilde{x}(t) \triangleq x(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k) \qquad (1.1)$$

Find the signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k) \qquad (1.2)$$



<Figure 1>

[Problem 2] (10 Points) Let $x(t_1, t_2)$ be a signal that depends upon two independent variables t_1 and t_2. The two-dimensional Fourier Transform of $x(t_1, t_2)$ is defined as

$$X(jw_1, jw_2) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2) e^{-j(w_1 t_1 + w_2 t_2)} dt_1 dt_2 \qquad (2.1)$$

(a) Determine the inverse transform of $X(jw_1, jw_2)$

(b) Determine the two-dimensional Fourier Transform of the following signal

$$x(t_1, t_2) \triangleq \begin{cases} e^{-|t_1|-|t_2|} & \text{for } 0 \le t_1 \le 1 \text{ and } 0 \le t_2 \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad (2.2)$$

[Problem 3] (10 Points) Find the discrete-time signal whose Fourier transform is given by

$$X(e^{j\omega}) = e^{-j\left(4\omega + \frac{\pi}{2}\right)} \frac{d}{d\omega}\left[\frac{2}{1+\frac{1}{4}e^{-j\left(\omega-\frac{\pi}{4}\right)}} + \frac{2}{1+\frac{1}{4}e^{-j\left(\omega+\frac{\pi}{4}\right)}}\right] \qquad (3.1)$$

[Problem 4] (10 Points) Consider a system built as the cascade interconnection of two LTI systems with frequency responses

$$H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{3}e^{-j\omega}} \tag{4.1}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega}} \tag{4.2}$$

(a) Find the difference equation describing the overall system.
(b) Determine the impulse response of the overall system.

[Problem 5] (20 Points) (a) Find the condition under which the formal Fourier transform of the signal

$$y[n] = \sum_{m=-\infty}^{n} x[m] \tag{5.1}$$

exists. Then, derive its formal Fourier transform.
(b) Show that the Fourier transform of the above signal not satisfying such a condition can be expressed as follows.

$$Y(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \tag{5.2}$$

(c) In Example 5.8 of the textbook, the Fourier transform $X(e^{j\omega})$ of the unit step $x[n] \triangleq u[n]$ is derived in the following steps.

First, the following facts are used.

$$g[n] \triangleq \delta[n] \xrightarrow{F} G(e^{j\omega}) = 1 \tag{5.3}$$

,

$$u[n] = \sum_{m=-\infty}^{n} g[m] \tag{5.4}$$

Then, applying the results in (5.1), (5.2) yields

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{(1 - e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \end{aligned} \tag{5.5}$$

Explain which parts in the above arguments are not mathematically logical.

[Problem 6] (20 Points) Let $x[n]$ be a signal that is 0 outside the interval $0 \le n \le N_1 - 1$. For $N \ge N_1$, the N-point DFT(discrete Fourier transform) of $x[n]$ is given by

$$\widetilde{X}[k] = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jk(2\pi/N)n}, \; k = 0, 1, \ldots, N-1 \tag{6.1}$$

It is convenient to write eq. (6.1) as

$$\widetilde{X}[k] = \frac{1}{N} \sum_{k=0}^{N-1} x[n] W_N{}^{nk}, \quad W_N \triangleq e^{-j2\pi/N} \tag{6.2}$$

Suppose that N is even. Let $f[n] = x[2n]$ represent the even-indexed samples of $x[n]$, and let $g[n] = x[2n+1]$ represent the odd-indexed samples.

(a) Show that $f[n]$ and $g[n]$ are zero outside the interval $0 \le n \le (N/2)-1$.

(b) Show that the N -point DFT $\widetilde{X}[k]$ of $x[n]$ can be expressed as

$$\begin{aligned}\widetilde{X}[k] &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk} + \frac{1}{N}W_N^k \sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk} \\ &= \frac{1}{2}\widetilde{F}[k] + \frac{1}{2}W_N^k \widetilde{G}[k], \quad k = 0,1,\ldots,N-1\end{aligned} \tag{6.3}$$

Where

$$\begin{aligned}\widetilde{F}[k] &\triangleq \frac{2}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk}, \\ \widetilde{G}[k] &\triangleq \frac{2}{N}\sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk}.\end{aligned} \tag{6.4}$$

(c) Show that, for all k,

$$\begin{aligned}\widetilde{F}[k+\frac{N}{2}] &= \widetilde{F}[k], \\ \widetilde{G}[k+\frac{N}{2}] &= \widetilde{G}[k].\end{aligned} \tag{6.5}$$

Note that $\widetilde{F}[k],\ k = 0,1,\ldots,(N/2)-1$, and $\widetilde{G}[k],\ k = 0,1,\ldots,(N/2)-1$, are the $(N/2)$ -point DFTs of $f[n]$ and $g[n]$, respectively. Thus, eq. (6.3) indicates that the length- N DFT of $x[n]$ can be calculated in terms of two DFTs of length $N/2$.

(d) Determine the number of complex multiplications required to compute $\widetilde{X}[k],\ k = 0,1,2,\ldots,N-1$, from eq. (6.3) by first computing $\widetilde{F}[k]$ and $\widetilde{G}[k]$. (Assume that $x[n]$ is complex and that the required values of W_N^{nk} have been pre-computed and stored in a table. For simplicity, do not exploit the fact that, for certain values of n and k, W_N^{nk} is equal to ± 1 or $\pm j$ and hence does not, strictly speaking, require a full complex multiplication. Also, ignore the multiplications by the quantity 1/2 in (6.3)).

- END -