Topics in Communications (Spring, 2008), Midterm

1. Inner and outer polyhedral approximations. Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and suppose that $x_1, \cdots, x_K$ are on the boundary of C. Suppose that for each i, $a_i^T(x - x_i) = 0$ defines a supporting hyperplane for C at x_i, *i.e.*, $C \subseteq \left\{ x | a_i^T(x - x_i) \leqq 0 \right\}$. Consider the two polyhedra

$$P_{inner} = conv\{x_1, \cdots,\ x_K\}$$

$$P_{outter} = \left\{ x | a_i^T(x - x_i) \leqq 0,\ i = 1, 2, \cdots, K \right\}$$

Show that $P_{inner} \subseteq C \subseteq P_{outer}$. Draw a picture illustrating this.

2. *Log-convexity of moment functions.* Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is nonnegative with $\mathbb{R}^+ \subseteq \operatorname{dom} f$. For $x \geqq 0$ define

$$\phi(x) = \int_0^\infty u^x f(u) du$$

Show that is a log-convex function. (If x is a positive integer, and f is a probability density function, then $\phi(x)$ is the xth moment of the distribution.)

Use this to show that the Gamma function,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du.$$

is log-convex for $x \geqq 1$.

3. *Network flow problem.* Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij} x_{ij}$, where c_{ij} are given constants. The total cost across the network is

$$C = \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} (usually assumed to be nonnegative) and an upper bound u_{ij}. The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_{ij}|$ ows out of the network. We assume that $1^T b = 0$, *i.e.*, the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero.

The problem is to minimize the total cost of ow through the network, subject to the constraints described above. Formulate this problem as an LP.

4. *Power assignment in a wireless communication system*. We consider n transmitters with powers $p_1,\cdots,p_n \geqq 0$, transmitting to n receivers. These powers are the optimization variables in the problem. We let $G \in \mathbb{R}^{n\times n}$ denote the matrix of path gains from the transmitters to the receivers; $G_{ij} \geqq 0$ is the path gain from transmitter j to receiver i.
The *signal power* at receiver i is then $S_i = G_{ii}p_i$, and the interference power at receiver i is $I_i = \sum_{i\neq k} G_{ik}p_k$. The signal to interference plus noise ratio, denoted SINR, at receiver i, is given by $S_i/(I_i+\sigma_i)$, where $\sigma_i > 0$ is the (self-) noise power in receiver i. The objective in the problem is to maximize the minimum SINR ratio, over all receivers, i.e., to maximize

$$\min_{i=1,\cdots,n} \frac{S_i}{I_i+\sigma_i}$$

There are a number of constraints on the powers that must be satisfied, in addition to the obvious one $p_i \geqq 0$. The first is a maximum allowable power for each transmitter, i.e., $p_i \leqq P_i^{\max}$, where $P_i^{\max} > 0$ is given. In addition, the transmitters are partitioned into groups, with each group sharing the same power supply, so there is a total power constraint for each group of transmitter powers. More precisely, we have subsets $K_1,\cdots,K_m$ of $\{1,\cdots,n\}$ with $K_1 \cup \cdots \cup K_m = \{1,\cdots,n\}$, and $K_i \cap K_j = \phi$ if $i \neq j$. For each group K_l, the total associated transmitter power cannot exceed $P_l^{gp} > 0$:

$$\sum_{k\in K_l} p_k \leqq P_l^{sp},\ l = 1,\cdots,\ m$$

Finally, we have a limit $P_k^{rc} > 0$ on the total received power at each receiver:

$$\sum_{k=1}^{n} G_{ik}p_k \leqq P_i^{rc},\ i = 1,\cdots,i$$

(This constraint reflects the fact that the receivers will saturate if the total received power is too large.)
Formulate the SINR maximization problem as a generalized linear-fractional program.

5. Hyperbolic constraints as SOC constraints. Verify that $x \in \mathbb{R}^n$, $y,z \in \mathbb{R}$ satisfy

$$x^T x \leqq yz,\ y \geqq 0,\ z \geqq 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y-z \end{bmatrix} \right\|_2 \leqq y+z,\ y \geqq 0,\ z \geqq 0$$

Use this observation to cast the following problems as SOCPs.
(a) maximizing harmonic mean

$$\text{maximize} \quad \left(\sum_{i=1}^{m} \frac{1}{a_i^T x - b_i} \right)^{-1}$$

with domain $\{x|Ax > b\}$, where a_i^T is ith row of A.

(b) maximizing geometric mean

$$\text{maximize} \left(\prod_{i=1}^{m} (a_i^T x - b_i) \right)^{1/m}$$

with domain $\{x|Ax > b\}$, where a_i^T is ith row of A.

Important Notice

A. Time: 2:30 ~ 4:30

B. Only textbook is allowed (No other books, documents, or papers)

C. Extra answer sheets will be used.