1 (40 points) Nonhomogeneous Boundary Conditions

If the ends $x = 0$ and $x = L$ of the bar are kept at constant temperatures U_1 and U_2, respectively, what is the temperature $u_I(x)$ in the bar after a long time (theoretically, as $t \to \infty$)? First guess, and answer the questions of this following PDE.

$$u_t = c^2 u_{xx} \quad 0 < x < L,\ t > 0$$

$$u(x, 0) = f(x),$$

$$u(0, t) = U_1,\ u(L, t) = U_2.$$

Hint. In order to solve the PDE, we need to set $u(x, t) = w(x, t) + v(x)$, where $v(x)$ satisfies that $v_t = v_{xx} = 0$ and $w(x, t)$ is a homogeneous solution of

$$w_t = c^2 w_{xx} \quad 0 < x < L,\ t > 0$$

$$w(0, t) = 0,\ w(L, t) = 0.$$

Solution: Since $v_t = v_{xx} = 0$ and $w(0, t) = u(0, t) - v(0) = 0,\ w(L, t) = u(L, t) - v(L) = 0$, we have $v(x) = c_1 x + c_2$ and

$$\begin{aligned} w(0, t) &= u(0, t) - c_2 \\ &= U_1 - c_2 = 0. \\ w(L, t) &= u(L, t) - (c_1 L + c_2) \\ &= U_2 - c_1 L - c_2 = 0. \end{aligned}$$

Solving the two equations,

$$v(x) = (U_2 - U_1)x/L + U_1$$

and

$$w(x, t) = u(x, t) - (U_2 - U_1)x/L - U_1.$$

Since $w(x, 0) = u(x, 0) - (U_2 - U_1)x/L - U_1 = f(x) - (U_2 - U_1)x/L - U_1$, we need only solve the problem

$$w_t = c^2 w_{xx} \quad 0 < x < L,\ t > 0$$

$$w(0, t) = 0,\ w(L, t) = 0.$$

$$w(x, 0) = f(x) - (U_2 - U_1))x/L - U_1.$$

And we know that

$$w(x, t) = \Sigma_{n=1}^{\infty} B_n\, sin\, \frac{n\pi x}{L} e^{-\lambda_n^2 t},\ \ (\lambda_n = cn\pi/L),$$

where

$$B_n = \frac{2}{L} \int_0^L [f(x) - (U_2 - U_1)x/L - U_1]\ sin\, \frac{n\pi x}{L} dx.$$

Since $lim_{t\to\infty} w(x, t) = 0$,

$$lim_{t\to\infty} u(x, t) = lim_{t\to\infty} w(x, t) + v(x) = v(x) = (U_2 - U_1)x/L + U_1.$$

2 (60 points)

1. (30 points)
$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2, \ 0 < y < 3$$
$$u(x,0) = sin\,(\pi\, x)$$
$$u(x,3) = sin\,(7\pi\, x/2), \quad u(0,y) = u(2,y) = 0$$

2. (30 points)
$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 2, \ 0 < y < 3,$$
$$\partial u/\partial \mathbf{n} = 0, \qquad y = 0 \ or \ y = 3,$$
$$\partial u/\partial \mathbf{n} = cos\,(\pi\, y) \quad x = 0, \ \partial u/\partial \mathbf{n} = cos\,(11\pi\, y/3) \ \ x = 2.$$

---

1. By separation of variables and $u(0,y) = u(2,y) = 0$,

$$u(x,y) = \Sigma_{n=0}^{\infty}\ sin\,\frac{n\pi x}{2}\left(c_n cosh\frac{n\pi y}{2} + d_n sinh\frac{n\pi y}{2}\right).$$

The boundary conditions give us

$$u(x,0) = sin\,(\pi\, x) = \Sigma_{n=0}^{\infty}\, c_n\, sin\,\frac{n\pi x}{2}$$

$$u(x,3) = sin\,(7\pi\, x/2) = \Sigma_{n=0}^{\infty}\ sin\,\frac{n\pi x}{2}\left(c_n cosh\frac{n\pi 3}{2} + d_n sinh\frac{n\pi 3}{2}\right).$$

Thus, only $c_2 = 1$ and $c_n = 0$, otherwise. And

$$c_7\, cosh\frac{21\pi}{2} + d_7 sinh\frac{21\pi}{2} = 1$$

and

$$c_2\, cosh\, 3\pi + d_2 sinh\, 3\pi = 0.$$

Thus $d_7 = (sinh\frac{21\pi}{2})^{-1}$, and $d_2 = -c_2 coth 3\pi$.
Therefore,

$$\begin{aligned} u(x,y) =&\ sin\,\pi\, x[c_2 cosh\,\pi\, y + d_2 sinh\,\pi\, y] + d_7\, sin\,\frac{7\pi x}{2} sinh\frac{7\pi y}{2} \\ =&\ sin\,\pi x[\ cosh\,\pi\, y - (coth\, 3\pi) sinh\,\pi\, y] + \left(sinh\frac{21\pi}{2}\right)^{-1}\ sin\,\frac{7\pi x}{2} sinh\frac{7\pi y}{2}. \end{aligned}$$

2. We know that

$$u_y(x,0) = u_y(x,3) = 0$$
$$u_x(0,y) = -cos\,(\pi\, y)$$

and

$$u_x(2,y) = cos\,(11\pi\, y/3).$$

By letting $u(x,y) = F(x)G(y)$, we have $G'' + p^2 G = 0$, $F'' - p^2 F = 0$ with $G'(0) = G'(3) = 0$ and $p_n = n\pi/3$. So, $G_0(y) = 1$, and $G_n(y) = cos\frac{n\pi\, y}{3}$. And also $F_0(x) = c_0 + d_0 x$ and $F_n(x) = \left(c_n cosh\frac{n\pi x}{3} + d_n sinh\frac{n\pi x}{3}\right)$. And the general solution

$$u(x,y) = c_0 + d_0 x + \Sigma_{n=1}^{\infty}\ cos\,\frac{n\pi y}{3}\left(c_n cosh\frac{n\pi x}{3} + d_n sinh\frac{n\pi x}{3}\right).$$

Finally, the boundary conditions determine the constants;

$$u_x(0,y) = -cos\,(\pi\, y)\ = d_0 + \Sigma_{n=1}^{\infty}\ \frac{n\pi}{3}\, d_n\, cos\, \frac{n\pi y}{3}.$$

$$u_x(2,y) = cos\,(\frac{11\pi\, y}{3}) = d_0 + \Sigma_{n=1}^{\infty} \frac{n\pi}{3}\ \ cos\, \frac{n\pi y}{3} \left(c_n sinh \frac{2n\pi}{3} + d_n cosh \frac{2n\pi}{3} \right).$$

Now, we know that $d_3 = -1/\pi$ and $d_n = 0$, otherwise.
And $c_3 = (coth\, 2\pi)/\pi$ and $c_{11} = \frac{3}{11\pi\, sinh(22\pi/3)}$. Therefore,

$$u(x,y) = c_0 + \frac{cos\,\pi\, y}{\pi}\,[-sinh\,\pi\, x + (coth\, 2\pi) cosh\,\pi\, x] + \frac{3}{11\pi\, sinh(22\pi/3)} cos \frac{11\,\pi\, y}{3} cosh \frac{11\pi\, x}{3}$$

3 (50 points) $u_t - u_{xx} = 5sinx$ with $u(0,t) = u(\pi,t) = 0$, and $u(x,0) = \frac{1}{2}\sin 3x - \frac{3}{7}\sin 11x$.

1. (10 points) Show that if $w(x)$ satisfies $w_{xx} = -5\sin x$ with $w(0) = w(\pi) = 0$ and $v_t - v_{xx} = 0$, $v(0,t) = v(\pi,t) = 0$ and $v(x,0) = f(x) - w(x)$.

   then $u(x,t) = v(x,t) + w(x)$ is the solution of the previous nonhomogeneous PDE in (1.1).

2. (10 points) Obtain $w(x)$ satisfying the previous two conditions.

3. (20 points) Obtain $v(x,t)$ which is a solution of the previous homogeneous heat equation in (1.2).

4. (10 points) Using (2) and (3) results, determine $u(x,t)$ for $x \in [0,\pi]$ and $t \geq 0$.

1.

$$u_t(x,t) - u_{xx}(x,t) = v_t(x,t) - v_{xx}(x,t) - w_{xx}(x) = 5sinx$$
$$u(0,t) = v(0,t) + w(0) = 0$$
$$u(\pi,t) = v(\pi,t) + w(\pi) = 0$$
$$u(x,0) = v(x,0) + w(x) = f(x) - w(x) + w(x) = f(x).$$

since $u(x,t) = v(x,t) + w(x)$, and $-w_{xx} = 5sinx$, $w(0) = w(\pi) = 0$, $v_t - v_{xx} = 0$, $v(0,t) = v(\pi,t) = 0$ and $v(x,0) = f(x)$.

2. After calculating w_{xx} with $w(x) = Asinx + Bcosx$, we know that $B = 0$ and $A = 5$. Thus,

$$w(x) = 5sinx$$

3. By solving the previous homogeneous heat equation of $v(x,t)$, we know that

$$v(x,t) = \sum_{n=1}^{\infty} B_n sin(nx) e^{-n^2 t}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2}\sin 3x - \frac{3}{7}\sin 11x - 5sinx\right) sin(nx)dx.$$

And the orthogonality of {sinx,sin2x, sin3x, sin4x, ..., sin(nx),... } makes us obtain that

$$v(x,t) = \left(\frac{1}{2}sin3x\right)e^{-3^2 t} - \left(\frac{3}{7}sin11x\right)e^{-11^2 t} - (5sinx)\,e^{-t},$$

4. Using 2 and 3 answers, for $x \in [0,\pi]$ and $t \geq 0$,

$$u(x,t) = v(x,t) + w(x) = \left(\frac{1}{2}sin3x\right)e^{-3^2 t} - \left(\frac{3}{7}sin11x\right)e^{-11^2 t} + 5sinx(1 - e^{-t}).$$

4 (50 points) Θ-Independent Dirichlet Problem on a Ball

Use the expression in spherical coordinates: $\Delta u = \frac{1}{\rho^2}\left[(\rho^2 u_\rho)_\rho + \frac{1}{sin\phi}(u_\phi sin\phi)_\phi + \frac{1}{sin^2\phi}u_{\theta\theta}\right]$

1. (30 points) Solve the θ-independent Dirichlet problem on a ball: $\Delta u \equiv \nabla^2 u = 0, \quad 0 < \rho < 1$, with $u(1,\theta,\phi) = f(\phi)$.
2. (20 points) Solve the Laplace equation inside the sphere $\rho = 1$ subject to the boundary condition: $u(1,\theta,\phi) = 3P_5(cos\phi) - 7P_2(cos\phi)$, where P_n is the n^{th} degree Legendre polynomial,

---

1. Since $\Delta u = 0$, we have $(\rho^2 u_\rho)_\rho + \frac{1}{sin\phi}(u_\phi sin\phi)_\phi = 0$, which, upon separation via $u = R(\rho)\Phi(\phi)$, becomes

$$\frac{[\rho^2 R'(\rho)]'}{R(\phi)} = -\frac{[\Phi'(\phi)sin\phi]'}{\Phi(\phi)sin\phi} = \lambda$$

So we get the two ODEs:

$$[\rho^2 R'(\rho)]' - \lambda R(\phi) = 0$$

$$[\Phi'(\phi)sin\phi]' + \lambda\Phi(\phi)sin\phi = 0.$$

Now, the substitution $x = cos\phi$ gives us the problem

$$[(1-x^2)\Phi'(\phi)]' + \lambda\Phi = 0, \quad -1 < x < 1$$

which is just Legendre's equation with $\lambda_n = n(n+1)$, and has $\Phi_n(x) = P_n(x)$, the n^{th} degree Legendre polynomial as the solution. And

$$\rho^2 R'' + 2\rho R' - n(n+1)R = 0,$$

and thus $R_n(\rho) = c_1\rho^n + c_2\rho^{-1-n}, \quad 0 < \rho < 1,\ n = 0, 1, 2, ..., .$
Since $0 < \rho < 1$, the general solution is

$$u(\rho,\phi,\theta) = \Sigma_{n=0}^{\infty}\ c_n\,\rho^n P_n(cos\phi).$$

Finally, the boundary condition gives us

$$u(1,\phi,\theta) = f(\phi) = \Sigma_{n=0}^{\infty}\ c_n\,P_n(cos\phi).$$

And again letting $x = cos\phi$, we have

$$f(cos^{-1}x) = \Sigma_{n=0}^{\infty}\ c_n\,P_n(x).$$

Thus, the c_n are just the Fourier-Legendre coefficients of the function $f(cos^{-1}x)$,

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(cos^{-1}x)P_n(x)dx = \frac{2n+1}{2}\int_0^\pi f(\phi)P_n(cos\,\phi)sin\,\phi\,d\phi$$

2. Since $f(\phi) = u(1,\theta,\phi) = 3P_5(cos\phi) - 7P_2(cos\phi)$,

$$u(\rho,\phi,\theta) = 3\rho^5 P_5(cos\phi) - 7\rho^2 P_2(cos\phi).$$