확률 2008년 중간시험2 - solution

1번 solution

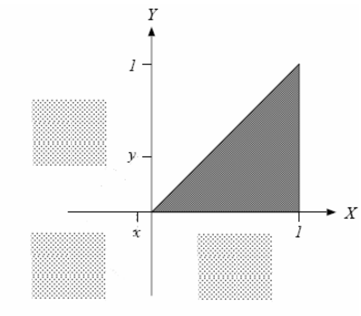
1. Random variables X and Y have joint PDF

$$f_{X,Y}(x,y)=\begin{cases}8xy & 0\le y\le x\le 1\\ 0 & otherwise\end{cases}$$

공통 채점 기준: 식 틀리면 0점, 식 맞고 답 틀리면 부분점수

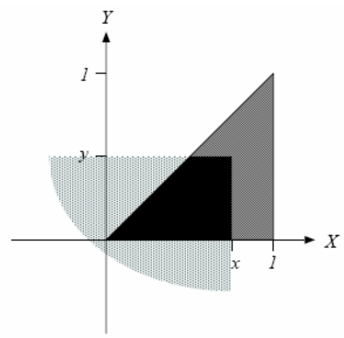
(1) Find the joint CDF $F_{X,Y}(x,y)$. (10 points)

a) 영역 $x<0\ \ or\ \ y<0$에 대해서



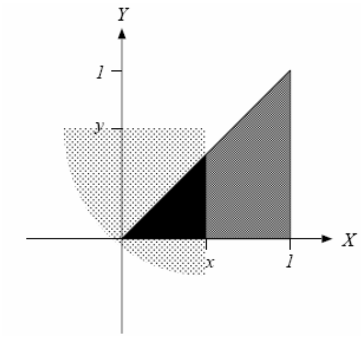
$$F_{X,Y}(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv=0$$ (2점)

b) 영역 $0\le y\le x\le 1$에 대해서



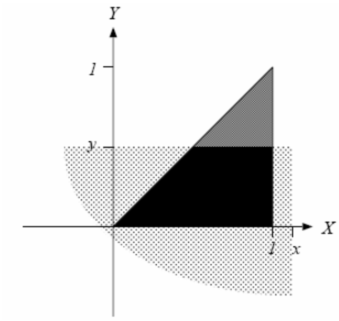
$$F_{X,Y}(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv$$
$$=\int_{0}^{y}\int_{v}^{x} 8uv\ dudv=2x^2y^2-y^4$$ (2점)

c) 영역 $0\le x\le y\ \ and\ \ 0\le x\le 1$에 대해서



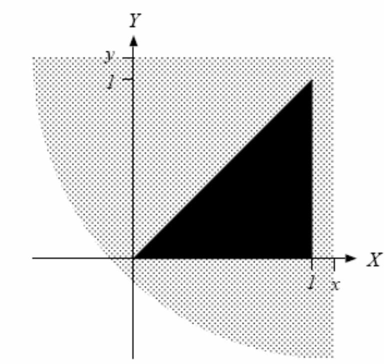
$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv$$
$$= \int_{0}^{x}\int_{0}^{u} 8uv\ dvdu = x^4$$ (2점)

d) 영역 $0 \le y \le 1$ *and* $x \ge 1$ 에 대해서



$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv$$
$$= \int_{0}^{y}\int_{v}^{1} 8uv\ dudv = 2y^2 - y^4$$ (2점)

e) 영역 $x > 1$ *and* $y > 1$ 에 대해서



$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv = 1$$ (2점)

a)~e)를 종합하면,

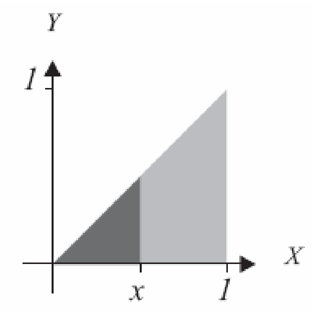
$$F_{X,Y}(x,y)=\begin{cases} 0 & x<0 \ \ or \ \ y<0 \\ 2x^2y^2-y^4 & 0\le y\le x\le 1 \\ x^4 & 0\le x\le y \ \ and \ \ 0\le x\le 1 \\ 2y^2-y^4 & 0\le y\le 1 \ \ and \ \ x\ge 1 \\ 1 & x>1 \ \ and \ \ y>1 \end{cases}$$

채점기준: 각 소문제(2점)당 식 틀리면 0점

식 맞고 답 틀리면 1점

(2) Find $F_X(x)$ and $F_Y(y)$. (10 points)

a) 영역 $0\le x<1$에 대해서 $F_X(x)$를 계산하면,

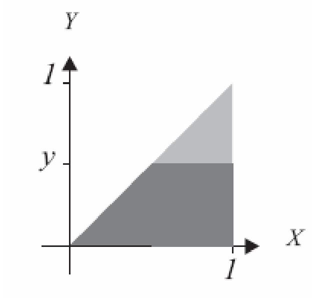


$$F_X(x)=\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dvdu=\int_{0}^{x}\int_{0}^{u} 8uv\ dvdu$$

$$=\int_{0}^{x} 4u^3du = x^4$$ (3점)

$$F_X(x)=\begin{cases} 0 & x<0 \\ x^4 & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$$ (2점: 범위 안 나누면 0점)

b) 영역 $0\le y<1$에 대해서 $F_Y(y)$를 계산하면,



$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv = \int_0^y \int_v^1 8uv\, dudv$$

$$= \int_0^y 4v(1-v^2)dv = 2y^2 - y^4$$ (3점)

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 2y^2 - y^4 & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$ (2점: 범위 안 나누면 0점)

채점기준: 식 틀리면 0점
식 맞고 답 틀리면 2점

(3) Find $Cov[X,Y]$. (10 points)

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 x(4x^3)dx = \frac{4}{5} = 0.8$$ (2점)

$$E[Y] = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^1 y(4y - 4y^3)dx = \frac{4}{3} - \frac{4}{5} = \frac{8}{15} = 0.5333$$ (2점)

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dydx = \int_0^1\int_0^u uv(8uv)dvdu$$

$$= \int_0^1 8u^2(\frac{1}{3}u^3)du = \frac{4}{9} = 0.4444$$ (3점)

$$Cov[XY] = E[XY] - E[X]E[Y] = \frac{4}{9} - \frac{4}{5}\cdot\frac{8}{15} = \frac{100-96}{225} = \frac{4}{225} = 0.0178$$ (3점)

채점기준: E[X],E[Y] 식 맞고 답 틀리면 1점
E[XY],Cov[XY] 식 맞고 답 틀리면 2점
식 틀리면 0점

(4) Find $Var[X+Y]$. (10 points)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2(4x^3)dx = \frac{2}{3}$$

$$Var[X] = E[X^2] - \{E[X]\}^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75} = 0.0267$$ (3점)

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_0^1 y^2(4y - 4y^3)dx = 1 - \frac{2}{3} = \frac{1}{3}$$

$$Var[Y] = E[Y^2] - \{E[Y]\}^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{75-64}{225} = \frac{11}{225} = 0.0489$$ (3점)

By theorem 4.15

$$Var[X+Y] = Var[X] + Var[Y] + 2Cov[XY]$$

$$= \frac{2}{75} + \frac{11}{225} + 2\frac{4}{225} = \frac{6+11+8}{225} = \frac{25}{225} = \frac{1}{9} = 0.1111$$ (4점)

채점기준: E[X], E[Y], Var[X+Y] 모두
식 맞고 답 틀리면 2점
식 틀리면 0점

2번 solution

(1)

$$p(k_1,k_2) = \sum_{k_3=0}^{n} \frac{n!}{k_1!k_2!k_3!(n-k_1-k_2-k_3)!} \cdot p_1^{k_1} p_2^{k_2} p_3^{k_3} (1-p_1-p_2-p_3)^{n-k_1-k_2-k_3}$$

$$= \frac{n!\,p_1^{k_1} p_2^{k_2}}{k_1!k_2!(n-k_1-k_2)!} \cdot \sum_{k_3=0}^{n} \frac{(n-k_1-k_2)!}{k_3!(n-k_1-k_2-k_3)!} \cdot p_3^{k_3} (1-p_1-p_2-p_3)^{n-k_1-k_2-k_3}$$

위 식에서 $\sum_{k_3=0}^{n} \frac{(n-k_1-k_2)!}{k_3!(n-k_1-k_2-k_3)!} \cdot p_3^{k_3} (1-p_1-p_2-p_3)^{n-k_1-k_2-k_3}$ 은 binomial theorem에 의해서 $\left[p_3 + (1-p_1-p_2-p_3)\right]^{n-k_1-k_2}$ 으로 변환할 수 있고 다음과 같이 간단하게 표현됨

$$p\left(k_2,k_3\mid k_1\right)=\frac{p\left(k_1,k_2,k_3\right)}{p\left(k_3\right)}=\frac{\left(n-k\right)!}{k_2!k_3!\left(n-k_1-k_2-k_3\right)!}\left(\frac{p_2}{1-p_1}\right)^{k_2}\left(\frac{p_3}{1-p_1}\right)^{k_3}\left(\frac{1-p_1-p_2-p_3}{1-p_1}\right)^{n-k_2-k_3}$$

$0\le k_2+k_3\le n-k_1$

$$\therefore P_{X_1,X_2,X_3}\left(k_2,k_3\mid k_1\right)=\frac{\left(n-k\right)!}{k_2!k_3!\left(n-k_1-k_2-k_3\right)!}\left(\frac{p_2}{1-p_1}\right)^{k_2}\left(\frac{p_3}{1-p_1}\right)^{k_3}\left(\frac{1-p_1-p_2-p_3}{1-p_1}\right)^{n-k_2-k_3}$$

$$p\left(k_1,k_2\right)=\frac{n!}{k_1!k_2!\left(n-k_1-k_2\right)!}p_1^{k_1}p_2^{k_2}\left(1-p_1-p_2\right)^{n-k_1-k_2}$$

$$\therefore P_{X_1,X_2}\left(k_1,k_2\right)=\frac{n!}{k_1!k_2!\left(n-k_1-k_2\right)!}p_1^{k_1}p_2^{k_2}\left(1-p_1-p_2\right)^{n-k_1-k_2}$$

where $k_1,k_2\ge 0\quad k_1+k_2\le n$

(2)

$$p\left(k_1\right)=\sum_{k_2=0}^{n}\frac{n!}{k_1!k_2!\left(n-k_1-k_2\right)!}p_1^{k_1}p_2^{k_2}\left(1-p_1-p_2\right)^{n-k_1-k_2}$$

Binomial theorem을 활용하면

$$\therefore P_{X_1}\left(k_1\right)=\frac{n!}{k_1!\left(n-k_1\right)!}p_1^{k_1}\left(1-p_1\right)^{n-k_1}$$

Where $k_1\ge 0\quad k_1\le n$

(3)

$$p\left(k_2,k_3\mid k_1\right)=\frac{p\left(k_1,k_2,k_3\right)}{p\left(k_3\right)}=\frac{\left(n-k\right)!}{k_2!k_3!\left(n-k_1-k_2-k_3\right)!}\left(\frac{p_2}{1-p_1}\right)^{k_2}\left(\frac{p_3}{1-p_1}\right)^{k_3}\left(\frac{1-p_1-p_2-p_3}{1-p_1}\right)^{n-k_2-k_3}$$

$$\therefore P_{X_1,X_2,X_3}\left(k_2,k_3\mid k_1\right)=\frac{\left(n-k\right)!}{k_2!k_3!\left(n-k_1-k_2-k_3\right)!}\left(\frac{p_2}{1-p_1}\right)^{k_2}\left(\frac{p_3}{1-p_1}\right)^{k_3}\left(\frac{1-p_1-p_2-p_3}{1-p_1}\right)^{n-k_2-k_3}$$

where $0\le k_2+k_3\le n-k_1$

3번 solution

3.(1)

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

(사칙연산정도)

계산실수 ; -1

나머지 ; -5

$$= \int_{0}^{\infty} e^{j\omega x} \cdot \lambda \cdot e^{-\lambda x} \, dx$$

$$= \int_{0}^{\infty} \lambda \cdot e^{(j\omega - \lambda)x} \, dx$$

$$= \lambda \left[-\frac{1}{\lambda - j\omega} \cdot e^{-(\lambda - j\omega)x} \right]_0^{\infty}$$

$$= \lambda \left(0 + \frac{1}{\lambda - j\omega} \right) = \frac{\lambda}{\lambda - j\omega}$$

(2)

$$G_N(z) = \sum_{k=-\infty}^{\infty} P_N(k)\, z^k$$

+1

$$= \sum_{k=0}^{\infty} p(1-p)^k \cdot z^k$$

$$= p \cdot \sum_{k=0}^{\infty} \{(1-p)z\}^k = \frac{p}{1-(1-p)z}$$

+4

(3)

$$\Phi_R(\omega) = E[e^{j\omega R}] = E[e^{j\omega(X_0+X_1+\cdots+X_n)}]$$

$$= E[e^{j\omega X_0}]\cdot\left(E[e^{j\omega X_1}]\cdots E[e^{j\omega X_n}]\right)$$

$$= E[e^{j\omega X}]\cdot E[e^{j\omega X}]^n$$

$$= \Phi_X(\omega)\cdot G_N(E[e^{j\omega X}])$$

$$= \Phi_X(\omega)\cdot G_N(\Phi_X(\omega)) \quad \leftarrow +6$$

$$= \left(\frac{\lambda}{\lambda - j\omega}\right)\cdot\frac{p}{1-(1-p)\dfrac{\lambda}{\lambda-j\omega}}$$

$$= \frac{\lambda p}{\lambda p - j\omega}$$

$$\therefore f_R(r) = \begin{cases} p\lambda e^{-p\lambda r} & r \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \leftarrow +4$$

(4)

i) $E[R] = \int_{-\infty}^{\infty} r \cdot f_R(r)\, dr$

$$= \int_{0}^{\infty} pre^{-pr}\, dr = \left[-re^{-pr} - \frac{1}{p}e^{-pr} \right]_{0}^{\infty}$$

$$= \frac{1}{p}$$

잘못구한 $f_R(r)$로 이용한경우 : -5

↳ 부분적분 과정이 나오지 않는 경우 형평성을 위해 ; -5

ii) $E[R^2] = \int_{0}^{\infty} pr^2 e^{-pr}\, dr$

$$= \left[r^2 e^{-pr} - \frac{2re^{-pr}}{p} + \frac{2 \cdot e^{-pr}}{-p^2} \right]_{0}^{\infty}$$

$$= \frac{2}{p^2}$$

부분적분 과정 ; -5

$Var[R] = E[R^2] - (E[R])^2$

$$= \frac{2}{p^2} - \frac{1}{p^2} = \frac{1}{p^2}$$

다른 방식 풀이

(3)

$$\Phi_R(\omega) = G_N(\Phi_X(\omega))$$

$$= \frac{p}{1-(1-p)\left(\frac{\lambda}{\lambda-j\omega}\right)}$$

$$= \frac{p(\lambda-j\omega)}{p\lambda-j\omega}$$ — +5

$$= p + (1-p)\frac{p\lambda}{p\lambda-j\omega}$$

$$\therefore f_R(r) = \begin{cases} p\delta(r) + (1-p)p\lambda e^{-p\lambda r} & , r \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$ +5

(4)

$$E[R] = \int_{-\infty}^{\infty} r \cdot f_R(r)\,dr = \int_0^{\infty} r\left\{p\delta(r) + (1-p)p\lambda e^{-p\lambda r}\right\}dr$$

(3)번 결과가 잘못되었을 경우 -5

E[R] -5

Var[R] -5

$$= \cancel{\int_0^{\infty} r p\delta(r)\,dr}^{\,0} + (1-p)\int_0^{\infty} r p\lambda \cdot e^{-p\lambda r}\,dr$$

$$= (1-p)\left\{\cancel{\left[r p\lambda \cdot \frac{1}{-p\lambda}e^{-p\lambda r}\right]_0^{\infty}}^{\,0} + \int_0^{\infty} p\lambda \frac{1}{p\lambda} e^{-p\lambda r}\,dr\right.$$

$$= (1-p)\left[-\frac{1}{p\lambda}e^{-p\lambda r}\right]_0^{\infty} = (1-p)\frac{1}{p\lambda}$$

$$= \frac{1-p}{p\lambda}$$

$\lambda = 1$ 이므로

$$= \frac{1-p}{p}$$ //

$$E[R^2] = \int_0^\infty r^2 \left\{ p\delta(r) + (1-p)p\lambda e^{-p\lambda r} \right\} dr$$

$$= (1-p)\int_0^\infty 2r e^{-p\lambda r}\, dr$$

$$= -\frac{2(1-p)}{(p\lambda)^2}$$

$$Var[R] = E[R^2] - (E[R])^2$$

$$= \frac{2(1-p)}{(p\lambda)^2} - \frac{(1-p)^2}{(p\lambda)^2} = \frac{(1-p)(2-1+p)}{(p\lambda)^2} = \frac{(1-p)(1+p)}{(p\lambda)^2}$$

$$= \frac{1-p^2}{(p\lambda)^2}$$

$\lambda = 1$ 이므로

$$= \frac{1-p^2}{p^2}$$

//

4번 solution

문제4.)

(a) Mean $[\bar{Y}] = [T][\bar{X}]$

[+10]

$$= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$$

+4

Covariance $[C_Y] = [T][C_X][T]^T$

$$= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 5 & \frac{-2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & 4 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-\frac{2}{\sqrt{5}} & \frac{9}{2}-\frac{5}{2\sqrt{5}} \\ \frac{9}{2}-\frac{5}{2\sqrt{5}} & \frac{21}{4}-\frac{2}{\sqrt{5}} \end{bmatrix}$$

+6

(b) Correlation Coefficient $C_{Y_1Y_2}$

[+5]

$C_{Y_1Y_2} = \rho\sigma_{Y_1}\sigma_{Y_2}$ 이므로

$$\rho = \frac{C_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}}$$ +2

$$= \frac{\frac{9}{2} - \frac{5}{2\sqrt{5}}}{\sqrt{6-\frac{2}{\sqrt{5}}}\sqrt{\frac{21}{4}-\frac{2}{\sqrt{5}}}}$$

$\fallingdotseq 0.7172$ +3

5번 solution

#5. (1). $F_{X,Y}(x,y) = \begin{cases} 1-e^{-(x+y)}, & x \geq 0,\ y \geq 0. \\ 0 & , \text{otherwise} \end{cases}$

joint CDF ?

→ 우선 $F_X(x)$, $F_Y(y)$를 찾아내보면,

$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise}. \end{cases}$

$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1, & y \geq 0 \\ 0, & \text{otherwise}. \end{cases}$

이로부터 어떤 $x \geq 0$, $y \geq 0$ 에 대하여도.

$P[X > x] = 0$. $P[Y > y] = 0$ 이 성립한다.

$x \geq 0$, $y \geq 0$ 에 대하여.

$\underline{P[\{X > x\} \cup \{Y > y\}]} \leq P[X > x] + P[Y > y] = 0$

$= 1 - P[X \leq x, Y \leq y] = 1 - (1 - e^{-(x+y)})$

$= e^{-(x+y)} \leq 0. \Rightarrow$ contradiction.

∴ 틀림.

(2). X, Y are independent & both exponentially distributed with mean one. $Z = X/Y$. then

$$f_Z(z) = \frac{1}{(1+z)^2}, \quad z > 0 \quad ?$$

$\Rightarrow$ exponential distribution 에서 $E[X] = \frac{1}{\lambda}$.

$E[X] = 1 = \frac{1}{\lambda}$ 로부터 $\lambda = 1$.

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \qquad f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Y를 $Y = y$로 가정하면,

$$f_Z(z|y) = |y| \cdot f_X(yz \,|\, y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y'| \cdot f_X(y'z \,|\, y') f_Y(y')\, dy'$$

$$= \int_{-\infty}^{\infty} |y'| \cdot f_{X,Y}(y'z, y')\, dy'$$

independent

$$= \int_0^{\infty} y' \cdot f_X(y'z) \cdot f_Y(y')\, dy' \quad z > 0.$$

$$= \int_0^{\infty} y' \cdot e^{-y'z} \cdot e^{-y'}\, dy'$$

$$= \frac{1}{(1+z)^2}, \quad z > 0.$$

$\therefore$ 맞음.

(3).

$$f_{X,Y} = \begin{cases} \frac{1}{2}, & -1 \leqq x \leqq y \leqq 1. \\ 0, & \text{otherwise}. \end{cases}$$

then $E[X|Y=y] = \frac{y-1}{2}$?

$(-1 \leq y \leq 1)$

$$\Rightarrow \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{1}{2}\cdot\int_{-1}^{y} dx = \frac{y+1}{2}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2}}{(y+1)/2} = \frac{1}{y+1}$$

$(-1 \leq x \leq y).$

$$E[X|Y=y] = \int_{-1}^{y} x \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_{-1}^{y} x \cdot \frac{1}{y+1}\,dx = \frac{1}{2(y+1)} \cdot x^2 \Big|_{-1}^{y}$$

$$= \frac{1}{2\cancel{(y+1)}} \cdot \cancel{(y+1)}(y-1) = \frac{y-1}{2}.$$

∴ 맞음.

(4). $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$Y = AX = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}.$$

$$P_X(\underline{x}) = \begin{cases} (1-p)p^{x_3} & x_1 < x_2 < x_3 \ ; \ x_1, x_2, x_3 \in \{1, 2, \cdots\} \\ 0. & \text{otherwise} \end{cases}$$

이면, $P_Y(\underline{y}) = \begin{cases} (1-p)^3 \cdot p^{(y_1+y_2+y_3)} & ; \ y_1, y_2, y_3 \in \{1, 2, \cdots\} \\ 0 & \text{otherwise} \end{cases}$

인가?

$\Rightarrow$. $P_Y(\underline{y}) = P[Y_1 = y_1, \ Y_2 = y_2, \ Y_3 = y_3]$

$= P[X_1 = y_1, \ X_2 - X_1 = y_2, \ X_3 - X_2 = y_3]$

$= P[X_1 = y_1 . \ X_2 = y_1 + y_2 . \ X_3 = y_1 + y_2 + y_3]$

$= (1-p) \cdot p^{x_3}$

$= (1-p) \cdot p^{(y_1+y_2+y_3)}$

$\therefore$ 틀림.

(5). $S_k = X_1 + X_2 + \cdots + X_k$.

X_i : independent. Poisson. $E[X_i] = \alpha_i$

then S_k is a poisson random variable

with mean $\alpha = \alpha_1 + \cdots + \alpha_k$?

$\rightarrow$ $G_{S_k}(z) = G_{X_1}(z) \cdot G_{X_2}(z) \cdots G_{X_k}(z)$

$= e^{\alpha_1(z-1)} \cdots e^{\alpha_k(z-1)}$

$= e^{(\alpha_1 + \cdots + \alpha_k)(z-1)}$

$\rightarrow$ S_k poisson with rate $\alpha_1 + \cdots + \alpha_k$.