Communication Systems

(Midterm Exam Solution(4/26))

1.

a) advantages: Using DPSK avoids the need for possibly complex carrier-recovery schemes to provide an accurate phase estimate.

disadvantages: The probability of error for DPSK is only slightly worse than ordinary BPSK, particularly at higher SNR.

b) **Bandwidth efficiency** refers to the amount of information that can be transmitted over a given bandwidth in a specific communication system. It is a measure of how efficiently a limited frequency spectrum is utilized by the physical layer protocol.

Energy efficiency는 동일한 Error probability에서 사용된 신호의 에너지 정도. Error probability에 대한 SNR($\frac{\mathcal{E}_b}{N_0}$)을 기준으로 효율성을 측정한다.

2.

a) Describe the Nyquist criterion for no ISI.

The necessary and sufficient condition for $x(t)$ to satisfy

$$x(nT) = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) \end{cases}$$

is that its Fourier transform X(f) satisfy

$$\sum_{m=-\infty}^{\infty} X(f + \frac{m}{T}) = T$$

b)

The modified duobinary signal, which is the special case to (approximately) physically realizable transmitting and receiving filters, is specified by the samples

$$x(\frac{n}{2W}) = x(nT) = \begin{cases} 1, & n = -1, \\ -1, & n = 1, \\ 0, & otherwise. \end{cases}$$

The corresponding pulse $x(t)$ is given by

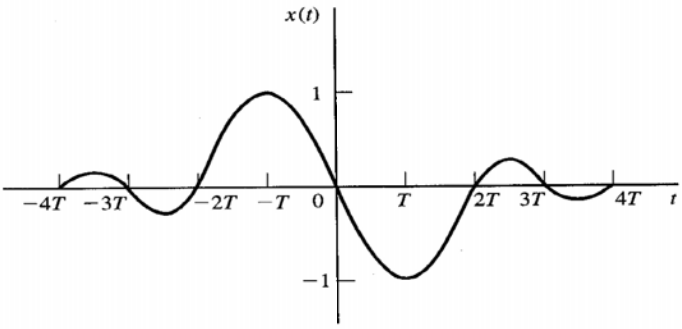
$$x(t) = \text{sinc}(\frac{t+T}{T}) - \text{sinc}(\frac{t-T}{T})$$

and its spectrum is given by

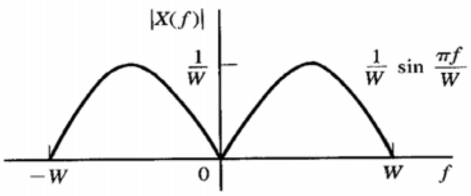
$$X(f) = \begin{cases} \frac{1}{2W} e^{j\frac{\pi f}{W}} - e^{j\frac{\pi f}{W}}, & |f| \leq W, \\ 0, & |f| > W, \end{cases}$$

$$= \begin{cases} \frac{1}{2W} \sin\frac{\pi f}{W}, & |f| \leq W, \\ 0, & |f| > W, \end{cases}$$

which is shown in the followed figures.



(a)



(b)

3.

For the two waveforms to be orthogonal, they must fulfill the orthogonality constraint of equation:

$$\int_0^T \cos(2\pi f_1 t + \phi)\cos 2\pi f_2 t \, dt = 0. \qquad (3.1)$$

Using the basic trigonometric identities, we can write equation (3.1) as

$$\cos\phi \int_0^T \cos 2\pi f_1 t \cos 2\pi f_2 t \, dt - \sin\phi \int_0^T \sin 2\pi f_1 t \sin 2\pi f_2 t \, d = 0, \qquad (3.2)$$

so that

$$\begin{aligned} &\cos\phi \int_0^T [\cos 2\pi (f_1 + f_2)t + \cos 2\pi (f_1 - f_2)t] \, dt \\ &\qquad - \sin\phi \int_0^T [\sin 2\pi (f_1 + f_2)t + \sin 2\pi (f_1 - f_2)t] \, d = 0. \end{aligned} \qquad (3.3)$$

Then, we can derive equation (3.4) as

$$\cos\phi \sin 2\pi (f_1 - f_2)T + \sin\phi [\sin 2\pi (f_1 - f_2)T - 1] \approx 0, \qquad (3.4)$$

For coherent detection, the spacings needed for orthogonality are found by setting $\phi = 0$, which gives

$$\sin 2\pi (f_1 - f_2)T = 0, \qquad (3.5)$$

or

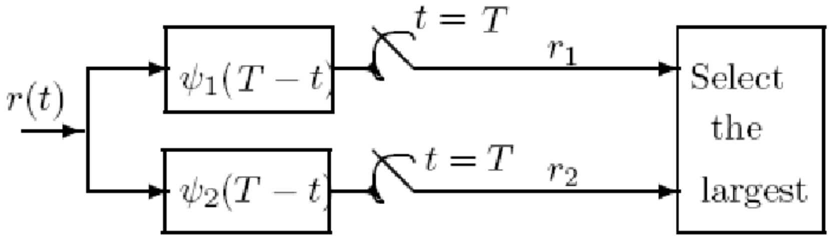
$$f_1 - f_2 = \frac{n}{2T} \qquad (3.6)$$

Thus the minimum spacing for coherent FSK signaling occurs for $n = 1$ as follows:

$$f_1 - f_2 = \frac{1}{2T} \qquad (3.7)$$

4.

a) The optimal receiver is shown in the next figure



$$\psi_1(t)=\frac{s_1(t)}{\sqrt{\varepsilon_b}}=\sqrt{\frac{3}{A^2T}}\frac{At}{T}(\because \varepsilon_b=\frac{A^2T}{3}), \tag{4.1}$$

$$\psi_2(t)=\frac{d_2(t)}{\sqrt{\varepsilon_2}}=\sqrt{\frac{4}{A^2T}}(A-\frac{3At}{2T})(\because \varepsilon_2=\frac{A^2T}{4}), \tag{4.2}$$

$$\begin{aligned} s_{11}&=\int_0^T s_1(t)\psi_1(t)dt=\sqrt{\varepsilon_b}=\sqrt{\frac{A^2T}{3}},\\ s_{12}&=\int_0^T s_1(t)\psi_2(t)dt=0,\\ s_{21}&=\int_0^T s_2(t)\psi_1(t)dt=\frac{1}{\sqrt{\varepsilon_b}}\frac{A^2T}{6}=\sqrt{\frac{A^2T}{12}},\\ s_{22}&=\int_0^T s_1(t)\psi_2(t)dt=\frac{1}{\sqrt{\varepsilon_2}}\frac{A^2T}{4}=\sqrt{\frac{A^2T}{4}}, \end{aligned} \tag{4.3}$$

$$\therefore \boldsymbol{r}=\boldsymbol{s}_m+\boldsymbol{n},\quad m=1,2, \tag{4.4}$$

where $\boldsymbol{r}=(r_1,\ r_2)$, $\boldsymbol{s}_m=(s_{m1},s_{m2})$, , and $\boldsymbol{n}=(n_1,n_2)$.

b) $\|\boldsymbol{s}_1\|^2=\|\boldsymbol{s}_2\|^2$ and two signals are equiprobable, so probability of error is

$$\begin{aligned} P_b&=\frac{1}{2}P(e\,|\,s_1)+\frac{1}{2}P(e\,|\,s_2)\\ &=\sqrt{\frac{d_{12}^2}{2N_0}}=\sqrt{\frac{A^2T}{6N_0}}=\sqrt{\frac{\varepsilon_b}{2N_0}}(\because d_{12}^2=\frac{A^2T}{3}). \end{aligned} \tag{4.5}$$

c) $\frac{\varepsilon_b}{N_0}$ (dB)= $\log_{10}\frac{\varepsilon_b}{N_0}$ =10dB, $\frac{\varepsilon_b}{N_0}$ =10.

$$\therefore\ P_b=Q\left(\sqrt{5}\right)=Q(0.236)=0.0091$$

5.

a) Gary code 를 사용한 M-ary PSK 에서는 error 가 발생한 k bit 의 한 symbol 당 한 개의 bit 만 error 가 발생하였다고 볼 수 있다.

$$\therefore P_b = \frac{1}{k} P_M$$

b) Symbol 한 개당 error probability = $\dfrac{P_M}{M-1} = \dfrac{P_M}{2^k - 1}$ $(\because k = \log_2 M)$

Error 가 발생한 한 개의 symbol 에서 error 발생 가능한 bit 수 = $\sum_{n=1}^{k} n \binom{k}{n} = k \times 2^{k-1}$

Bit error probability $P_b = \dfrac{1}{k} \times (k \times 2^{k-1}) \times \dfrac{P_M}{2^k - 1} = \dfrac{2^{k-1}}{2^k - 1} P_M \approx \dfrac{P_M}{2}$ $(k \gg 1)$

6.

The channel bandwidth is W =4000Hz.

a)-c): Pulse shape has $\alpha = 0.5$, so $\frac{1}{2T}(1+\alpha) = 2000$. $\frac{1}{T}$ =2667.

a) Binary PSK : 2667 bps.

b) 4-ary PSK : 5334 bps.

c) 16-ary rectangular QAM :10668 bps

d) binary orthogonal FSK, with noncoherent detection:

The frequency spacing between twofrequencies $\triangle f = \frac{1}{T}$. Then, the two frequencies are $f_c + \frac{1}{2T}$ and $f_c - \frac{1}{2T}$. Since w =4000 Hz, $\frac{1}{2T}$ =1000. Hence, the bit rate is 2000 bps.

e) orthogonal 4-ary FSK with noncoherent detection:

The frequency spacing between twofrequencies $\triangle f = \frac{1}{T}$. Then, the four frequencies are $f_c + \frac{3}{2T}$, $f_c + \frac{1}{2T}$, $f_c - \frac{1}{2T}$,and $f_c - \frac{3}{2T}$. $\frac{1}{2T}$ =500Hz, so the symbol rate is $\frac{1}{T}$ =1000 and since each symbol carries two bits of information, the bit rate is 2000bps.

7.

a)

$$\psi_1(t)=\begin{cases}\sqrt{\frac{2}{T}}, & 0\le t<\frac{T}{2}\\ 0, & otherwise\end{cases}\quad \psi_2(t)=\begin{cases}\sqrt{\frac{2}{T}}, & \frac{T}{2}\le t<T\\ 0, & otherwise\end{cases}$$

b) Assuming that the signal $s_1(t)$ is transmitted, the received vector at the output of the samplers is

$$\boldsymbol{r}=\left[\sqrt{\frac{A^2T}{2}}+n_1,\ n_2\right], \tag{7.1}$$

where n_1, n_2 are zero mean Gaussian random variables with variance $\frac{N_0}{2}$. The probability of error $P(\varepsilon\,|\,s_1)$ is

$$\begin{aligned}P(\varepsilon\,|\,s_1)&=P(n-2-n_1>\sqrt{\frac{A^2T}{2}})\\ &=\frac{1}{\sqrt{2\pi N_0}}\int_{\frac{A^2T}{2}}^{\infty}e^{-\frac{x^2}{2N_0}}dx=Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]\end{aligned} \tag{7.2}$$

Where we have used the fact the $n=n_2-n_1$ is a zero-mean Gaussian random variable with variance N_0. So that

$$P(\varepsilon)=\frac{1}{2}P(\varepsilon\,|\,s_1)+\frac{1}{2}P(\varepsilon\,|\,s_2)=Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \tag{7.3}$$

c) The signal waveform $\psi_1(\frac{T}{2}-t)$ matched to $\psi_1(t)$ is exactly the same with the signal waveform $\psi_2(T-t)$ matched to $\psi_2(t)$. That is,

$$\psi_1(\frac{T}{2}-t)=\psi_2(T-t)=\psi_1(t)=\begin{cases}\sqrt{\frac{2}{T}}, & 0\le t<\frac{T}{2}\\ 0, & otherwise\end{cases} \tag{7.4}$$

d) If the signal $s_1(t)$ is transmitted, then the received signal $r(t)$ is

$$r(t)=s_1(t)+\frac{1}{2}s_1(t-\frac{T}{2})+n(t) \tag{7.5}$$

$$\begin{aligned}r_1&=A\sqrt{\frac{2}{T}}\frac{T}{2}+n_1=2\sqrt{\frac{A^2T}{8}}+n_1\\ r_2&=\frac{A}{2}\sqrt{\frac{2}{T}}\frac{T}{2}+n_2=\sqrt{\frac{A^2T}{8}}+n_2\end{aligned} \tag{7.6}$$

If the optimal receiver uses a threshold V to base its decisions, that is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} V \tag{7.7}$$

Then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > \sqrt{\frac{A^2T}{8}} - V) = Q\left[\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] \tag{7.8}$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2(t - \frac{T}{2}) + n(t) \tag{7.9}$$

$$\begin{aligned} r_1 &= n_1 \\ r_2 &= A\sqrt{\frac{2}{T}}\frac{T}{2} + n_2 = \sqrt{\frac{A^2T}{2}} + n_2 \end{aligned} \tag{7.10}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \sqrt{\frac{A^2T}{2}} + V) = Q\left[\sqrt{\frac{A^2T}{2N_0}} + \frac{V}{\sqrt{N_0}}\right] \tag{7.11}$$

Thus, the average probability of error is given by

$$P(e) = \frac{1}{2}Q\left[\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2}Q\left[\sqrt{\frac{A^2T}{2N_0}} + \frac{V}{\sqrt{N_0}}\right] \tag{7.12}$$

$$\frac{\partial P(e)}{\partial V} = 0, \ \therefore V = -\frac{1}{2}\sqrt{\frac{A^2T}{8}} \tag{7.13}$$

e) Let a be fixed to some value between 0 and 1. Then we obtain

$$\begin{aligned} P(e|s_1,\alpha) &= P(n_2 - n_1 > (1-\alpha)\sqrt{\frac{A^2T}{2}} - V(\alpha)) = Q\left[(1-\alpha)\sqrt{\frac{A^2T}{2N_0}} - \frac{V(\alpha)}{\sqrt{N_0}}\right] \\ P(e|s_2,\alpha) &= P(n_1 - n_2 > \sqrt{\frac{A^2T}{2}} + V(\alpha)) = Q\left[\sqrt{\frac{A^2T}{2N_0}} + \frac{V(\alpha)}{\sqrt{N_0}}\right] \end{aligned} \tag{7.14}$$

$$P(e|\alpha) = \frac{1}{2}P(e|s_1,\alpha) + \frac{1}{2}P(e|s_2,\alpha) \tag{7.15}$$

$$\frac{\partial P(e)}{\partial V} = 0, \ \therefore V(\alpha) = -\frac{\alpha}{2}\sqrt{\frac{A^2T}{2}} \tag{7.16}$$

$$V = \int_0^1 V(\alpha) f(\alpha) da = -\frac{1}{2}\sqrt{\frac{A^2T}{2}}\int_0^1 \alpha d\alpha = -\frac{1}{4}\sqrt{\frac{A^2T}{2}} \tag{7.17}$$

8.

The output of the matched filter at the time instant mT is

$$y_m = \sum_k a_m x_{k-m} + v_m = a_m + \frac{1}{4} a_{m-1} + v_m. \tag{8.1}$$

The autocorrelation function of the noise sample v_m is

$$E\left[v_k v_j \right] = \frac{N_0}{2} x_{k-j}. \tag{8.2}$$

Thus, the variance of the noise is

$$\sigma_v^2 = \frac{N_0}{2} x_0 = \frac{N_0}{2} \tag{8.3}$$

If a symbol by symbol detector is employed and we assume that the symbols $a_m = a_{m-1} = \sqrt{\varepsilon_b}$ have been transmitted, then the probability of error

$$\begin{aligned} P(\varepsilon \mid a_m = a_{m-1} = \sqrt{\varepsilon_b}) &= P(y_m < 0 \mid a_m = a_{m-1} = \sqrt{\varepsilon_b}) \\ &= P(v_m < -\frac{5}{4}\sqrt{\varepsilon_b}) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\varepsilon_b}} e^{-\frac{v_m^2}{N_0}} dv_m \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\frac{2\varepsilon_b}{N_0}}} e^{-\frac{v^2}{2}} dv = Q\left[\frac{5}{4}\sqrt{\frac{2\varepsilon_b}{N_0}} \right] \end{aligned} \tag{8.4}$$

If however $a_{m-1} = -\sqrt{\varepsilon_b}$, then

$$P(\varepsilon \mid a_m = \sqrt{\varepsilon_b},\, a_{m-1} = -\sqrt{\varepsilon_b}) = P(\frac{3}{4}\sqrt{\varepsilon_b} + v_m < 0) = Q\left[\frac{3}{4}\sqrt{\frac{2\varepsilon_b}{N_0}} \right] \tag{8.5}$$

Since the two symbols $\sqrt{\varepsilon_b}$, $-\sqrt{\varepsilon_b}$ are used with equal probability, we conclude that

$$\begin{aligned} P(e) &= P(\varepsilon \mid a_m = \sqrt{\varepsilon_b}) = P(\varepsilon \mid a_m = -\sqrt{\varepsilon_b}) \\ &= \frac{1}{2} Q\left[\frac{5}{4}\sqrt{\frac{2\varepsilon_b}{N_0}} \right] + \frac{1}{2} Q\left[\frac{3}{4}\sqrt{\frac{2\varepsilon_b}{N_0}} \right] \end{aligned} \tag{8.6}$$