Quiz #1 Solution

1. $y'' + 9y = 18x + 36\sin 3x$

i) y_h

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$\therefore y_h = c_1\cos 3x + c_2\sin 3x$

ii) $y_{p_1} = C_1 x + C_2$, $y'_{p_1} = C_1$, $y''_{p_1} = 0$

$9C_1 x = 18x$

$\therefore C_1 = 2$

$y_{p_1} = 2x$

iii) $y_{p_2} = Kx\cos 3x + Mx\sin 3x$

$y'_{p_2} = K\cos 3x - 3Kx\sin 3x + M\sin 3x + 3Mx\cos 3x$

$y''_{p_2} = (-9Mx - 6K)\sin 3x + (-9Kx + 6M)\cos 3x$

$y''_{p_2} + 9y_{p_2} = \cancel{9Kx\cos 3x} + \cancel{9Mx\sin 3x} + (-\cancel{9Kx} + 6M)\cos 3x + (-\cancel{9Mx} - 6K)\sin 3x$

$= 6M\cos 3x - 6K\sin 3x = 36\sin 3x$

$\therefore M = 0,\ K = -6$

$\therefore \underline{\underline{y = y_h + y_{p_1} + y_{p_2} = C_1\cos 3x + C_2\sin 3x + 2x - 6x\sin 3x}}$

2. $(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$

$\frac{1}{F}dF(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{-e^{x+y} - ye^y}{e^{x+y} + ye^y} = -1$

$\therefore$ $\underline{\underline{\text{Integrating factor } F(y) = e^{\int(-1)dy} = e^{-y}}}$

3. (1) $y' + y\tan x = \sin 2x$. $y(0) = 1$

$y' + p(x)y = r(x)$

$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + c\right)$

$p(x) = \tan x$, $r(x) = \sin 2x$

$\therefore y = e^{-\int \tan x dx}\left(\int e^{\int \tan x dx} \sin 2x dx + c\right)$

$= e^{\ln \cos x}\left(\int e^{\ln(1/\cos x)} \sin 2x dx + c\right)$

$= \cos x\left(\int \frac{2\sin x\cos x}{\cos x} dx + c\right) = \cos x\left(\int 2\sin x dx + c\right)$

$= C_1\cos x - 2\cos^2 x$

initial condition 대입

$y(0) = C_1 - 2 = 1 \quad \therefore C_1 = 3$

$\underline{\underline{\therefore y = 3\cos x - 2\cos^2 x}}$

(2) $y'' + 4y' + 5y = 0$. $y(0) = 2$ $y'(0) = -5$

$y = ce^{\lambda x}$

$\lambda^2 + 4\lambda + 5 = 0$

$\therefore \lambda = -2 \pm i \Rightarrow y = e^{-2x}(A\cos x + B\sin x)$

$y' = -2e^{-2x}(A\cos x + B\sin x) + e^{-2x}(B\cos x - A\sin x)$

$= e^{-2x}(B - 2A)\cos x + e^{-2x}(-2B - A)\sin x$

initial condition 대입

$y(0) = C_1 = 2$

$y'(0) = C_2 - 2C_1 = -5 \quad \therefore C_2 = -1$

$\underline{\underline{\therefore y = e^{-2x}(2\cos x - \sin x)}}$

3.(3) $x^2y + 3xy' + y = 0$. $y(1) = 4$. $y'(1) = -2$

Euler - Cauchy Eq'n

$x^2y'' + axy' + by = 0 \;\leftarrow y = x^m$

$\Rightarrow m^2 + (a-1)m + b = 0$

$a=3,\; b=1$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$\therefore m = -1$

$y = (C_1 + C_2 \ln x)x^{-1}$

$y' = \frac{C_2}{x^2} + (C_1 + C_2 \ln x)\left(-\frac{1}{x^2}\right) = -\frac{C_1}{x^2} + C_2(1 - \ln x)\frac{1}{x^2}$

initial condition 대입

$y(1) = C_1 = 4$

$y'(1) = -C_1 + C_2 = -2 \quad \therefore C_2 = 2$

$\therefore \underline{\underline{y = (4 + 2\ln x)x^{-1}}}$

(4) $y' + (x+1)y = e^{x^2}y^3$. $y(0) = 0.5$

$y' + p(x)y = g(x)y^a \;\longleftarrow u(x) = [y(x)]^{1-a}$

$\Rightarrow u'(x) + (1-a)pu = (1-a)g(x)$

$p = x+1,\; g = e^{x^2},\; a = 3$

$u' - 2(x+1)u = -2e^{x^2}$

$u(x) = e^{-\int -2(x+1)dx}\left[e^{\int -2(x+1)dx}(-2e^{x^2})dx + C\right] = e^{x^2+2x}\left[\int e^{-x^2-2x}(-2e^{x^2})dx + C\right]$

$= e^{x^2+2x}\left[-2\int e^{-2x}dx + C\right]$

$= e^{x^2+2x}(e^{-2x} + C) = e^{x^2}(1 + Ce^{2x})$

$u(x) = [y(x)]^{-2}$ 이므로

initial condition $u(0) = [y(0)]^{-2} = 4 \quad (\because y(0) = \frac{1}{2})$

$u(0) = 1 + C = 4 \quad \therefore C = 3$

$\therefore \underline{\underline{u(x) = e^{x^2}(1 + 3e^{2x})}}$

4. $y'' + y = \sec x = \dfrac{1}{\cos x}$

i) y_h

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$\therefore y_h = A\cos x + B\sin x$

ii) y_p

Using Method of Variation of Parameters

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$, $y_1 = \cos x$, $y_2 = \sin x$

$W = \cos^2 x + \sin^2 x = 1$

$y_p = -y_1 \int \dfrac{y_2 r}{W} dx + y_2 \int \dfrac{y_1 r}{W} dx = -\cos x \int \dfrac{\sin x}{\cos x} dx + \sin x \int \dfrac{\cos x}{\cos x} dx$

$= -\cos x \int \tan x \, dx + x\sin x = \cos x \ln|\cos x| + x\sin x$

$$\therefore \underline{\underline{y = y_h + y_p = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x}}$$

5. $(x^2D^2 + 6xD + 6I)y = x^2$

$x^2 y'' + 6xy' + 6y = x^2$

i) y_h

$x^2 y'' + 6xy' + 6y = 0$

Euler - Cauchy Eq'n

$y = x^m$. $m^2 + (a-1)m + b = 0$, $a = 6$. $b = 6$

$m^2 + 5m + 6 = 0$

$m = -2, -3$

$\therefore y_h = c_1 x^{-2} + c_2 x^{-3}$

5. ii) y_p

① Using Method of Undetermined Coefficients

$y_p = kx^2 + mx + n$, $y_p' = 2kx + m$. $y_p'' = 2k$

$x^2(2k) + 6x(2kx + m) + 6(kx^2 + mx + n) = x^2$

$20kx^2 + 12mx + 6n = x^2$

$\therefore k = \frac{1}{20}$, $m = 0$, $n = 0$

$\therefore y_p = \frac{1}{20}x^2$

② Using Method of Variation of Parameters

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$. $y_1 = x^{-2}$, $y_2 = x^{-3}$, $r(x) = 1$

$W = x^{-2}(-3x^{-4}) - x^{-3}(-2x^{-3})$

$= -3x^{-6} + 2x^{-6} = -x^{-6}$

$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

$= -x^{-2} \int \frac{x^{-3}}{-x^{-6}} dx + x^{-3} \int \frac{x^{-2}}{-x^{-6}} dx$

$= -x^{-2} \int -x^3 dx + x^{-3} \int (-x^4) dx$

$= x^{-2} \frac{x^4}{4} - x^{-3} \frac{x^5}{5} = \frac{1}{20}x^2$

$\therefore y = y_h + y_p = c_1 x^{-2} + c_2 x^{-3} + \frac{1}{20}x^2$