QuizⅠ solution

1. Using the definition of the gradient, prove that $\dfrac{dV}{dl} = (\nabla V)\bullet\hat{l}$, $\hat{l}$ denote the unit vector of $\vec{l}$

Sol)

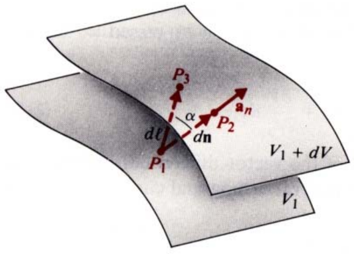


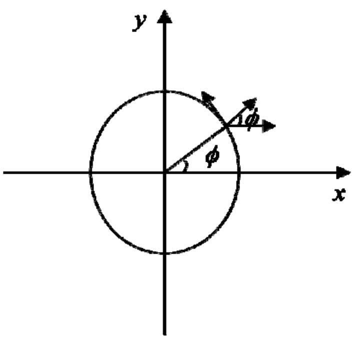

**FIGURE 2–24**
Concerning gradient of a scalar.

$$\operatorname{grad} V = \nabla V \triangleq a_n \frac{dV}{dn} = \hat{n}\frac{dV}{dn}$$

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{n}\cdot\hat{l} = (\nabla V)\cdot\hat{l}$$

2. Prove that $\dfrac{\partial\hat{r}}{\partial\phi} = \hat{\phi}$, $\dfrac{\partial\hat{\phi}}{\partial\phi} = -\hat{r}$

Sol)




$$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi,\quad \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\frac{\partial\hat{r}}{\partial\phi} = \frac{\partial}{\partial\phi}(\hat{x}\cos\phi + \hat{y}\sin\phi) = \hat{x}(-\sin\phi) + \hat{y}\cos\phi = -\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi}$$

$$\frac{\partial\hat{\phi}}{\partial\phi} = \frac{\partial}{\partial\phi}(-\hat{x}\sin\phi + \hat{y}\cos\phi) = -\hat{x}\cos\phi + \hat{y}(-\sin\phi) = -(\hat{x}\cos\phi + \hat{y}\sin\phi) = -\hat{r}$$

3. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dz'}{\left(r^2 + z'^2\right)^{3/2}} = \frac{2}{r^2}$

Sol)

$$z' = r\tan\theta \rightarrow dz' = r\sec^2\theta\, d\theta$$

$$\int_{-\pi/2}^{\pi/2} \frac{r\sec^2\theta}{\left(r^2 + r^2\tan^2\theta\right)^{3/2}} d\theta = \int_{-\pi/2}^{\pi/2} \frac{r\sec^2\theta}{r^3(1+\tan^2\theta)^{3/2}} d\theta = \frac{1}{r^2}\int_{-\pi/2}^{\pi/2} \frac{1}{\sec\theta} d\theta = \frac{1}{r^2}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{1}{r^2}\sin\theta\,\Big|_{-\pi/2}^{\pi/2} = \frac{2}{r^2}$$

4. Show that $\nabla\left(\frac{1}{R}\right) = -\frac{\hat{R}}{R^2}$ where $R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

Sol)

$$\nabla\left(\frac{1}{R}\right) = \hat{x}\frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \hat{y}\frac{\partial}{\partial y}\left(\frac{1}{R}\right) + \hat{z}\frac{\partial}{\partial z}\left(\frac{1}{R}\right)$$

$$= \hat{x}\left(-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}\cdot 2(x-x')\right)$$

$$+ \hat{y}\left(-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}\cdot 2(y-y')\right)$$

$$+ \hat{z}\left(-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}\cdot 2(z-z')\right)$$

$$= -\frac{1}{R^3}(\hat{x}(x-x')+\hat{y}(y-y')+z(z-z'))$$

$$= -\frac{1}{R^2}\frac{\hat{x}(x-x')+\hat{y}(y-y')+z(z-z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = -\frac{\hat{R}}{R^2}$$