1st Exam Solution

1. Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius a. The axes of the wires are separated by a distance D.

Sol)

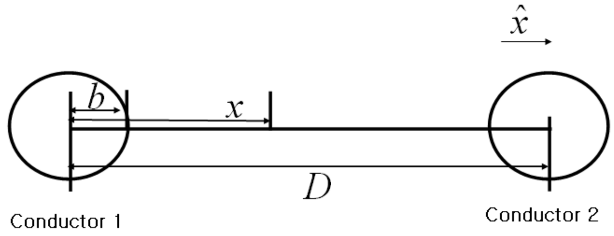
assuming d>>a, $\rho_a = \rho_l$, $\rho_b = -\rho_l$ in free space $\varepsilon = \varepsilon_0$

$$\bar{E}_a = a_r \frac{\rho_a}{2\pi\varepsilon r}, \quad \bar{E}_b = a_r \frac{\rho_b}{2\pi\varepsilon r}$$

$$V_{ab} = -\int_D^a \bar{E}_a \cdot dl - \int_a^D \bar{E}_b \cdot dl = -\int_D^a (a_r \frac{\rho_l}{2\pi\varepsilon r}) \cdot (a_r dr) - \int_D^a (a_r \frac{\rho_l}{2\pi\varepsilon r}) \cdot (a_r dr)$$

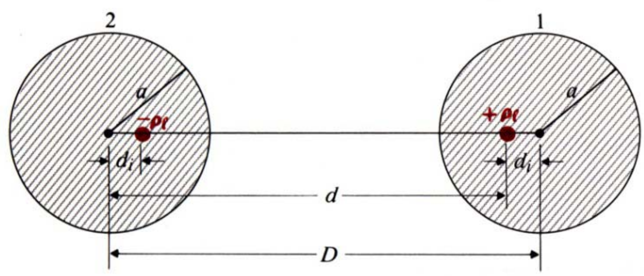
$$= \frac{\rho_l}{\pi\varepsilon} \ln(r)\Big|_a^D = \frac{\rho_l}{\pi\varepsilon} \ln\frac{D}{a}$$

$$C = \frac{Q}{V_{ab}} = \frac{\rho_l \cdot L}{\frac{\rho_l}{\pi\varepsilon}\ln\frac{D}{a}} = \frac{\pi\varepsilon L}{\ln\frac{D}{a}}$$



$\therefore$ The capacitance per unit length $\dfrac{C}{L} = \dfrac{\pi\varepsilon}{\ln\frac{D}{a}}$ [F/m]

Using the method of image. (not assuming d>>a)



assuming $\rho_1 = \rho_l$, $\rho_2 = -\rho_l$

$$\bar{E}_1 = a_r \frac{\rho_1}{2\pi\varepsilon_0 r}, \quad \bar{E}_2 = a_r \frac{\rho_2}{2\pi\varepsilon_0 r}$$

$$V_1 = -\int_d^a \bar{E}_a \cdot dl = -\frac{\rho_l}{2\pi\varepsilon_0} \ln\frac{a}{d}$$

$$V_2 = -\int_a^d \bar{E}_b \cdot dl = \frac{\rho_l}{2\pi\varepsilon_0} \ln\frac{a}{d}$$

The capacitance per unit length is

$$\mathrm{C} = \frac{\rho_l}{V_1 - V_2} = \frac{\pi\varepsilon_0}{\ln(d/a)}$$

where

$$d = D - d_i = D - \frac{a^2}{d} \text{ from which we obtain } d = \frac{1}{2}(D + \sqrt{D^2 - 4a^2})$$

$$\therefore \ \mathrm{C} = \frac{\pi\varepsilon_0}{\ln[(D/2a) + \sqrt{(D/2a)^2 - 1}]} \text{ or } \frac{\pi\varepsilon_0}{\cosh^{-1}(D/2a)} \text{ [F/m]}$$

2. A metal sphere of radius a has a uniform surface charge distribution ρ_s. The permittivity of the surrounding region varies as $\varepsilon = \varepsilon_0\left(1 + a/r\right)$.

Find (a) $\overline{D}, \overline{E}$ and $\overline{P}$ everywhere in space, (b) the bound charge densities, and (c) the energy density, and (d) find the potential in the dielectric region.

Sol)

(a) spherical symmetry → Gauss law 적용

$$\overline{D} = \hat{r}D_r, \qquad d\vec{s} = \hat{r}ds$$

$$Gauss\,law: \oint_S \overline{D} \bullet d\vec{s} = Q$$

$$\oint_S \overline{D} \bullet d\vec{s} = D_r \int_S ds = D_r 4\pi r^2,\ Q = \int_S \rho_s\, ds = \rho_s 4\pi a^2$$

$$\therefore \overline{D} = \hat{r}\frac{a^2\rho_s}{r^2}\ [C/m^2]$$

$$\overline{D} = \varepsilon_0\overline{E},\ \therefore \overline{E} = \hat{r}E_r = \hat{r}\frac{a^2\rho_s}{\varepsilon r^2} = \hat{r}\frac{a^2\rho_s}{\varepsilon_0 r(a+r)}\ [V/m]$$

$$\overline{P} = \overline{D} - \varepsilon_0\overline{E}, \therefore \overline{P} = \hat{r}P_r = \hat{r}(D_r - \varepsilon_0 E_r) = \hat{r}\frac{a^3\rho_s}{r^2(a+r)}\ [C/m^2]$$

(b)

$$\rho_{sb} = \overline{P} \bullet \hat{n}$$

$$\therefore \rho_{sb}\big|_{r=a} = \overline{P} \bullet (-\hat{r}) = -\frac{a^3\rho_s}{a^2(a+a)} = -\frac{\rho_s}{2}\ [C/m^2]$$

$$\rho_{sb}\big|_{r=b>a} = \overline{P} \bullet \hat{r} = \frac{a^3\rho_s}{b^2(a+b)}\ [C/m^2]$$

$$\rho_{vb} = -\nabla \bullet \overline{P}$$

$$\therefore \rho_{vb} = -\frac{1}{r^2}\frac{\partial}{\partial r}r^2 P_r = -\frac{1}{r^2}\frac{\partial}{\partial r}\frac{a^2\rho_s}{(a+r)} = \frac{a^3\rho_s}{r^2(a+r)^2}\ [C/m^3]$$

(c)

$$w_e = \frac{1}{2}\overline{D} \bullet \overline{E} = \frac{1}{2}\varepsilon E^2 = \frac{D^2}{2\varepsilon}$$

$$\therefore w_e = \frac{1}{2}\left(\hat{r}\frac{a^2\rho_s}{r^2}\right) \bullet \left(\hat{r}\frac{a^2\rho_s}{\varepsilon_0 r(a+r)}\right) = \frac{1}{2}\frac{a^4\rho_s^2}{r^3(a+r)^2}\ [J/m^3]$$

(d)

$$V = -\int_\infty^r \overline{E} \bullet d\vec{l}$$

$$\therefore V = -\int_\infty^r \frac{a^2\rho_s}{\varepsilon_0 r(a+r)}dr = \frac{-a^2\rho_s}{\varepsilon_0}\int_\infty^r \frac{1/a}{r} - \frac{1/a}{r+a}dr = \frac{-a\rho_s}{\varepsilon_0}\left(\int_\infty^r \frac{1}{r}dr - \int_\infty^r \frac{1}{r+a}dr\right)$$

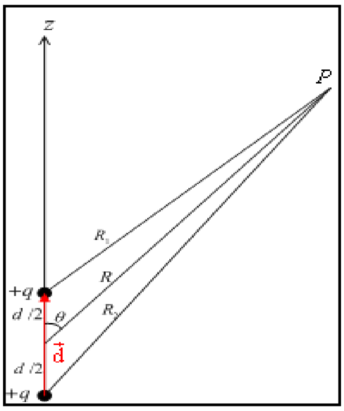
$$= \frac{a\rho_s}{\varepsilon_0}\left(\ln\frac{r+a}{r}\right)\Bigg|_\infty^r = \frac{a\rho_s}{\varepsilon_0}\ln\left(1 + \frac{a}{r}\right)[V]$$

3. Show that the magnitude of the electric field intensity of an electric dipole is $E = \frac{p}{4\pi\varepsilon_0 R^3}\left[1+3\cos^2\theta\right]^{1/2}$

Sol)

1. Find electric field intensity of an electric dipole

$\langle$method 1$\rangle$ find $\vec{E}$ directly



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\vec{R_1}}{|\vec{R_1}|^3} - \frac{\vec{R_2}}{|\vec{R_2}|^3}\right\} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\vec{R}-\frac{\vec{d}}{2}}{\left|\vec{R}-\frac{\vec{d}}{2}\right|^3} - \frac{\vec{R}+\frac{\vec{d}}{2}}{\left|\vec{R}+\frac{\vec{d}}{2}\right|^3}\right\}$$

$$\left|\vec{R}-\frac{\vec{d}}{2}\right|^{-3} = \left[\left(\vec{R}-\frac{\vec{d}}{2}\right)\bullet\left(\vec{R}-\frac{\vec{d}}{2}\right)\right]^{-3/2} = \left[R^2 - \vec{R}\bullet\vec{d} + \frac{d^2}{4}\right]^{-3/2}$$

$$= R^{-3}\left[1 - \frac{\vec{R}\bullet\vec{d}}{R^2} + \frac{d^2}{4R^2}\right]^{-3/2} \cong R^{-3}\left[1 - \frac{\vec{R}\bullet\vec{d}}{R^2}\right]^{-3/2} \cong R^{-3}\left[1 + \frac{3}{2}\frac{\vec{R}\bullet\vec{d}}{R^2}\right]$$

Similarly, $\left|\vec{R}+\frac{\vec{d}}{2}\right|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\vec{R}\bullet\vec{d}}{R^2}\right]$

$$\therefore \vec{E} \cong \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\vec{R}\bullet\vec{d}}{R^2}\vec{R} - \vec{d}\right] = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\vec{R}\bullet\vec{p}}{R^2}\vec{R} - \vec{p}\right] \quad (\text{let, } \vec{p} = q\vec{d})$$

$$\left(\text{using } \vec{p} = \hat{z}p = p\left(\hat{R}\cos\theta - \hat{\theta}\sin\theta\right) \Rightarrow \vec{R}\bullet\vec{p} = Rp\cos\theta\right)$$

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{R}2\cos\theta + \hat{\theta}\sin\theta\right)$$

$\langle$method 2$\rangle$ find $\vec{E}$ from V

$$V = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{R_1} - \frac{1}{R_2}\right\}$$

$$\frac{1}{R_1} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left[1 + \frac{d}{2R}\cos\theta\right]$$

Similarly, $\frac{1}{R_2} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left[1 - \frac{d}{2R}\cos\theta\right]$

$$\therefore V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{\vec{p}\bullet\hat{R}}{4\pi\varepsilon_0 R^2} \quad (\text{let, } \vec{p} = q\vec{d})$$

$$\vec{E} = -\vec{\nabla}V = -\hat{R}\frac{\partial V}{\partial R} - \hat{\theta}\frac{\partial V}{R\partial\theta}$$

$$\therefore \vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{R}2\cos\theta + \hat{\theta}\sin\theta\right)$$

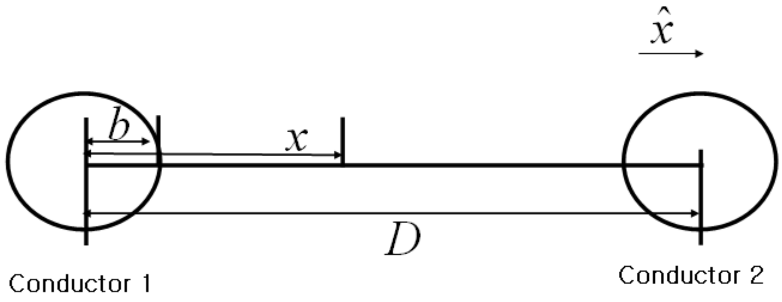
2. Calculate the magnitude

$$\left|\vec{E}\right| = \sqrt{E_r^2 + E_\theta^2} = \frac{p}{4\pi\varepsilon_0 R^3}\sqrt{(2\cos\theta)^2 + (\sin\theta)^2}$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}\sqrt{4\cos^2\theta + 1 - \cos^2\theta} = \frac{p}{4\pi\varepsilon_0 R^3}\left[1+3\cos^2\theta\right]^{1/2}$$

4. The conductors of an isolated two-wire transmission line, each of radius b, are spaced at a distance D apart. Assuming $D \gg b$ and a voltage V_0 between the lines, find the force per unit length on the lines.

Sol)



Assume line charge densities ρ_l, $-\rho_l$ on conductors 1 and 2 respectively. At any point x between to conductors, electric field intensity is

$$\overline{E} = \hat{x}\frac{\rho_l}{2\pi\varepsilon_0 x} - \hat{x}\frac{-\rho_l}{2\pi\varepsilon_0 (D-x)} = \hat{x}\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{x} + \frac{1}{D-x}\right)$$

Then electric potential V between conductor 1 and 2 is,

$$V_{12} = V_1 - V_2 = -\int_{x=D-b}^{x=b} \overline{E} \cdot d\overline{x} = -\int_{D-b}^{b} \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{x} + \frac{1}{D-x}\right)dx$$

$$= -\frac{\rho_l}{2\pi\varepsilon_0}\int_{D-b}^{b} \frac{1}{x} + \frac{1}{D-x}dx = -\frac{\rho_l}{2\pi\varepsilon_0}\cdot\left(\ln x - \ln(D-x)\right)\Big|_{D-b}^{b}$$

$$= -\frac{\rho_l}{2\pi\varepsilon_0}\cdot\left(\ln\frac{b}{D-b} - \ln\frac{D-b}{b}\right) = -\frac{\rho_l}{\pi\varepsilon_0}\ln\frac{b}{D-b} = \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{D-b}{b}$$

$$= \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{D(1-{b}/{D})}{b} \approx \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{D}{b} \qquad (D >> b)$$

And capacitance per unit length between two-wire transmission line(distance D) is

$$C = \frac{\rho_l}{V_{12}} = \frac{\pi\varepsilon_0}{\ln {D}/{b}} \qquad (F/m)$$

Now using principle of virtual displacement with fixed potential V_0,

$$\overline{F} = \nabla W_e = \hat{x}\frac{\partial}{\partial x}\left(\frac{1}{2}CV^2\right) = \hat{x}\frac{V_0^2}{2}\frac{\partial}{\partial x}\left(\frac{\pi\varepsilon_0}{\ln {x}/{b}}\right) = -\hat{x}\frac{\pi\varepsilon_0 V_0^2}{2x\left(\ln {x}/{b}\right)^2}$$

in this case, the distance between the wires is x=D,

$$\therefore \overline{F} = -\hat{x}\frac{\pi\varepsilon_0 V_0^2}{2D\left(\ln {D}/{b}\right)^2}$$