2. 과제 목록

과제 없음

3. 시험문제 및 시험 정답

(1) 1차 중간고사

Introduction to Random Variables and Random Processes, Exam 1, 2008.3.20.

Name		Student number	

Total 100 points: 1:(1)10, (2)10, 2:(1)10, (2)10, (3)15, (4)15, 3:(1)10, (2)20

1. (1) List the probability axioms. (2) Prove the following statement using only the axioms. If $A \subset B$, then $P(A) \leq P(B)$.

2. There are four cities, A, B, C, and D, and there are five roads between them: road 1 between A and B, road 2 between B and C, road 3 between C and D, road 4 between D and A, and road 5 between A and C. When it snows at night, each road has the probability p of being blocked by snow, independently of any other road. Consider the probability P_{AB} that city A is accessible from B. (1) Define an appropriate sample space and a probability allocation function on it. (2) Determine the event that A is accessible from B. (3) Find P_{AB}. (4) Are the event that A is accessible from B and the event that C is accessible from D independent? Explain why.

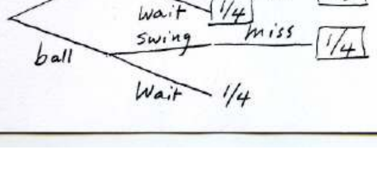
3. Consider a simplified baseball game where the pitcher throws a "strike" and "ball" with equal probability. The batter swings at the ball with probability 0.5. If he swings at a "strike" ball, he has the equal probability of hit and miss, but if he swings at a "ball" ball, he always misses. To make the problem simpler, let the out count be two "strikes" (ie, if the batter gets two "strikes", then he is out of the game), and let the batter walk to the first base at three "balls". If the batter swings at a ball and misses, then it is counted as a "strike". There are three cases for the batter to stop batting: (a) he makes a hit, (b) he gets two "strikes", or (c) he gets three "balls". (1) Find the probability that the batter gets a "strike". (2) Find the pmf of the number of balls the pitcher pitches for one batter.

Intro to rv and rp, Exam 1 solution, 2008. 3. 20.

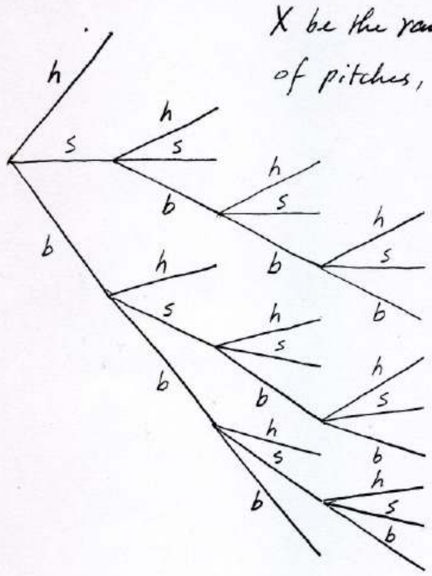
1. (1) Axiom 1: For any event A, $P(A) \geq 0$, Axiom 2: $P(S) = 1$. Axiom 3: For any countable collection $A_1, A_2, \cdots$ of mutually exclusive events, $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$. (2) If $A \subset B$, we can express B as $A \cup (B-A)$. Since A and $(B-A)$ are mutually exclusive, $P(B) = P(A) + P(B-A)$ by Axiom 3. From Axiom 1, $P(B-A) \geq 0$, and hence $P(A) \leq P(B)$.

2. (1) If we let 0 and 1 denote "blocked" and "accessible", respectively, for each road, an overnight snow will result in a five bit sequence. That is, 00110 represents that road 1, 2, and 5 are blocked and road 3 and 4 are okay. A reasonable sample space is the set of all five bit binary sequences: {00000, 00001, ⋯, 11111}.
(2) {01001, 01011, 01101, 01110, 01111, and 16 sequences beginning with 1} $\boxed{\text{The prob alloc fcn for 01100 is } p^3(1-p)^2 \text{, for example.}}$
(3) $P(10000, 10001, \cdots, 11111) = 1-p$, $P(01001, 01011, 01101, 01111) = p(1-p)^2$, and $P(01110) = p^2(1-p)^3$. So $P_{AB} = (1-p) + p(1-p)^2 + p^2(1-p)^3$. (4) By symmetry $P_{CD} = P_{AB}$.
The prob that both AB and CD are accessible = P(road 1 and 3 are okay) + $P(01011, 01101, 01110, 01111, 10011, 11010, 11011) = (1-p)^2 + 5p^2(1-p)^3 + 2p(1-p)^4$
Since $P_{AB}P_{CD} = [(1-p) + p(1-p)^2 + p^2(1-p)^3]^2 \neq$
they are not independent. Note the left side is a polynomial of degree 10 while the right side is of degree 5.

3. For each ball thrown, we can draw this tree diagram. (1) So the batter gets a "strike" for the three cases indicated by boxes. $P(\text{strike}) = \frac{1}{8} + \frac{1}{4} + \frac{1}{4} = \frac{5}{8}$



(2) From the diagram above, we have three results for each pitch: a hit with prob 1/8, a "strike" with prob 5/8, and a "ball" with prob 1/4. For the repeated throws of balls, now, we can draw another tree diagram as this, where h, s, b denotes hit, strike, ball, respectively. By letting X be the random variable representing the number of pitches, we can tell



$$P_X(1) = 1/8$$

$$P_X(2) = \frac{5}{8}\cdot\frac{1}{8} + \frac{5}{8}\cdot\frac{5}{8} + \frac{1}{4}\cdot\frac{1}{8} = 1/2$$

$$P_X(3) = \frac{5}{8}\cdot\frac{1}{4}\cdot\left(\frac{1}{8}+\frac{5}{8}\right) + \frac{1}{4}\cdot\frac{5}{8}\left(\frac{1}{8}+\frac{5}{8}\right) + \frac{1}{4}\cdot\frac{1}{4}\cdot\left(\frac{1}{8}+\frac{1}{4}\right) = \frac{33}{128}$$

$$P_X(4) = \frac{5}{8}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{5}{8}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{5}{8} = \frac{15}{128}$$

and $P_X(x) = 0$ for $x \neq 1, 2, 3,$ or 4

(2) 2차 중간고사

Introduction to Random Variables and Random Processes, Exam 2, 2008.4.10.

Total 100 points: 1:(1)11, (2)11, (3)12, 2:(1)9, (2)11, (3)8, 3:(1)13, (2)12, (3)13

1. A student is watching the night sky with a telescope. Let X be a random variable modeling the number of stars observable through the telescope fixed at one position (1) Find the pmf best suited for X. (2) Find the mean of X. (3) Find the variance of X.

2. Let the sample space be $[0, 1]$ and the probability allocation function be $q(s) = as$, where a is a constant and q is the function that assigns probability to each sample point such that for an event A, $P(A) = \int_A q(s)ds$. Let $X(s) = 1 - s^2$. (1) Find a. (2) Find the cdf of X. (3) Find the pdf of X.

3. In a school, the morning classes end between noon and 12:30 pm with uniform distribution, and the afternoon classes begin at 1 pm. It takes 50 minutes for a student to eat lunch at the cafeteria, but at 12:50 pm, when they find they cannot finish it by 1 pm, half of the students stop eating and leave, while the rest continue until 1 pm. Let X denote the random variable representing the length of the time a student eats lunch. (1) Find the pdf of X. (2) Find the EX. (3) Find $var(X)$.

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Intro to rvs and rps, Exam 2, 2008. 4. 10.

1. (1) Poisson pmf $p_X(x) = \begin{cases} \frac{\alpha^x}{x!} e^{-\alpha}, & x = 0, 1, 2, \dots \\ 0, & \text{else} \end{cases}$

(2) $EX = \sum_{x=0}^{\infty} x\, p_X(x) = \sum_{x=1}^{\infty} \frac{\alpha^x}{(x-1)!} e^{-\alpha} \overset{y = x-1}{=} \sum_{y=0}^{\infty} \alpha \cdot \frac{\alpha^y}{y!} e^{-\alpha} = \alpha$

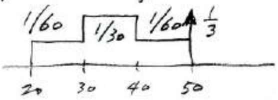
can change to 1

(3) $Var(X) = EX^2 - \alpha^2$, $EX^2 = \sum_{x=0}^{\infty} x^2 p_X(x) = \sum_{x=0}^{\infty} [x(x-1) + x]\, p_X(x)$
$= \sum_{x=2}^{\infty} \frac{\alpha^x}{(x-2)!} e^{-\alpha} + EX = \sum_{y=0}^{\infty} \alpha^2 \frac{\alpha^y}{y!} e^{-\alpha} + \alpha = \alpha^2 + \alpha$, $Var(X) = \alpha$.

2. (1) $\int_S g(s)\, ds = 1$. So $a \int_0^1 s\, ds = \frac{a}{2} = 1$, $a = 2$. (2) $F_X(x) = P(X \le x)$
$= P(1 - s^2 \le x) = \begin{cases} P(S \ge \sqrt{1-x}), & 0 \le x < 1 \\ 0, & x < 0 \\ 1, & x \ge 1 \end{cases}$ ① $= \int_{\sqrt{1-x}}^{1} 2s\, ds = x$

$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$ (3) $f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{else.} \end{cases}$

3. If the morning classes end before 12:10 pm, a student can finish his/her lunch. So $P(X = 50) = \int_0^{10} \frac{1}{30}\, dx = 1/3$. For the remaining probability 2/3, half of the students will take 20 to 40 minutes, uniformly, if they decide to leave early, and the other half 30 to 50 if they remain to continue eating. (1) $f_X(x) = \frac{1}{3}\delta(x - 50) + \frac{1}{3} f_1(x) + \frac{1}{3} f_2(x)$, where $f_1(x)$ and $f_2(x)$ are uniform pdf over $[20, 40]$ and $[30, 50]$, respectively.



(2) $EX = \int x f_X(x)\, dx = \int_{20}^{30} \frac{x}{60}\, dx + \int_{30}^{40} \frac{x}{30}\, dx + \int_{40}^{50} \frac{x}{60}\, dx + \frac{50}{3} = \frac{1}{120}(30^2 - 20^2) + \frac{1}{60}(40^2 - 30^2)$
$+ \frac{1}{120}(50^2 - 40^2) + \frac{50}{3} = \frac{1}{120}(-20^2 - 30^2 + 40^2 + 50^2) + \frac{50}{3} = 40$

(3) $Var(X) = EX^2 - (EX)^2$. $EX^2 = \int_{20}^{30} \frac{x^2}{60}\, dx + \int_{30}^{40} \frac{x^2}{30}\, dx + \int_{40}^{50} \frac{x^2}{60}\, dx + \frac{50^2}{3}$
$= \frac{1}{180}(-20^3 - 30^3 + 40^3 + 50^3) + \frac{50^2}{3} = \frac{15200}{9}$

$Var(X) = \frac{15200}{9} - 40^2 = \frac{800}{9}$

(3) 3차 중간고사

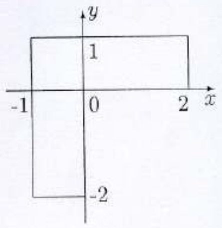
Introduction to Random Variables and Random Processes, Exam 3, 2008.5.6.

Total 100 points: 1:(1)12, (2)8; 2:(1)6, (2)12, (3)12, (4)10; 3:(1)8, (2)12, (3)12, (4)8

1. For a constant $a > 0$, random variables X and Y have the joint pdf $f_{X,Y}(x,y) = \begin{cases} 1/a^2 & 0 \le x \le a, 0 \le y \le a \\ 0 & \text{otherwise} \end{cases}$. Let $W = \max\left(\frac{X}{Y}, \frac{Y}{X}\right)$. (1) Find the cdf of W. (2) Find the pdf of W.

2. Suppose you arrive at a bus stop at time 0 and at the end of each minute, with probability p, a bus arrives, or with probability $1-p$, no bus arrives. Whenever a bus arrives, you board that bus with probability q and depart. Let T equal the number of minutes you stand at a bus stop. Let N be the number of buses that arrive while you wait at the bus stop. (1) Identify the set of points (n,t) for which $p_{N,T}(n,t) > 0$. (2) Find the conditional pmfs $p_{T|N}(t|n)$. (3) Find $p_{N,T}(n,t)$. (4) Find the marginal pmfs $p_N(n)$ and $p_T(t)$.

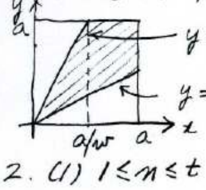
3. Random variables X and Y have the uniform joint pdf over the region in the figure. Let $Z = \min(X,Y)$. (1) Find $f_{X|Y}(x|y)$. (2) Find $E(X|Y)$. (3) Find $F_Z(z)$ and plot it. (4) Find $f_Z(z)$ and plot it.



Intro to rvs and rps, Exam 3, 2008. 5. 6

1. (1) $F_W(w) = P(\max(\frac{X}{Y}, \frac{Y}{X}) \le w) = P(\frac{X}{Y} \le w, \frac{Y}{X} \le w)$

$= P(Y \ge X/w, Y \le wX) = \begin{cases} \frac{1}{a^2}(a^2 - \frac{a^2}{w}) = (w-1)/w, & w \ge 1 \\ 0, & \text{otherwise.} \end{cases}$



(2) $f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/w^2, & w \ge 1 \\ 0, & \text{otherwise} \end{cases}$

2. (1) $1 \le n \le t$ (2) Given $N=n$, $n-1$ buses arrived in $t-1$ minutes, and the nth bus arrived after t minutes. $p_{T|N}(t|n) = \begin{cases} \binom{t-1}{n-1} p^{n-1}(1-p)^{t-n} p, & t \ge n \\ 0, & \text{otherwise} \end{cases}$

$= \begin{cases} \binom{t-1}{n-1} p^n (1-p)^{t-n}, & t \ge n \\ 0, & \text{otherwise} \end{cases}$ (3) $p_{T,N}(t,n) = p_{T|N}(t|n)\, p_N(n)$

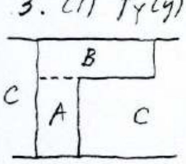
$= \begin{cases} \binom{t-1}{n-1} p^n (1-p)^{t-n} (1-q)^{n-1} q, & 1 \le n \le t \\ 0, & \text{otherwise} \end{cases}$ (4) $p_N(n) = \begin{cases} (1-q)^{n-1} q, & n = 1, 2, \cdots \\ 0, & \text{otherwise.} \end{cases}$

$p_T(t) = \begin{cases} (1-pq)^{t-1} pq, & t = 1, 2, \cdots \\ 0, & \text{otherwise} \end{cases}$ ← This can be obtained by considering the experiment as independent Bernoulli trials with success prob pq. Note also that

$$\sum_{n=1}^{t} \binom{t-1}{n-1} p^n (1-p)^{t-n} (1-q)^{n-1} q = \underbrace{\sum_{j=0}^{k} \binom{k}{j} \left(\frac{p(1-q)}{1-pq}\right)^j \left(1 - \frac{p(1-q)}{1-pq}\right)^{k-j}}_{=1} (1-pq)^k pq,$$

where $k = t-1$ and $j = n-1$.

3. (1) $f_Y(y) = \begin{cases} 1/5, & -2 \le y < 0 \\ 3/5, & 0 \le y < 1 \\ 0, & \text{else} \end{cases}$ $f_{X|Y}(x|y) = \begin{cases} 1, & (x,y) \in A \\ 1/3, & (x,y) \in B \\ 0, & (x,y) \in C \\ \text{undefined}, & \text{elsewhere} \end{cases}$

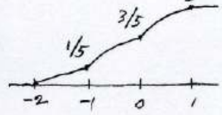


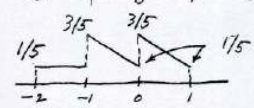
(2) $E(X|Y=y) = \begin{cases} -1/2, & -2 \le y < 0 \\ 1/2, & 0 \le y < 1 \\ \text{undefined}, & \text{else} \end{cases}$ $E(X|Y) = \begin{cases} -1/2, & \text{with prob } 2/5 \\ 1/2, & \text{with prob } 3/5 \end{cases} \triangleq Z$

That is, $p_Z(z) = \begin{cases} 2/5, & z = -1/2 \\ 3/5, & z = 1/2 \end{cases}$

(3). $F_Z(z) = P(X \le z \text{ or } Y \le z) = 1 - P(X > z, Y > z) = \begin{cases} 0, & z < -2 \\ (z+2)/5, & -2 \le z < -1 \\ 3/5 + z(1-z)/5, & -1 \le z < 0 \\ 1 - (2-z)(1-z)/5, & 0 \le z < 1 \\ 1, & z \ge 1 \end{cases}$

(4) $f_Z(z) = \begin{cases} 1/5, & -2 \le z < -1 \\ (1-2z)/5, & -1 \le z < 0 \\ (3-2z)/5, & 0 \le z < 1 \\ 0, & \text{else} \end{cases}$





(4) 4차 중간고사

Introduction to Random Variables and Random Processes, Exam 4, 2008.5.27.

Total 100 points: 1:(1)14, (2)11, (3)9; 2:(1)12, (2)10, (3)7; 3:(1)10, (2)10, (3)7, (4)10

1. Let $X_1, X_2, \cdots, X_n$ denote iid random variables with the common cdf $F(x)$ and pdf $f(x)$. (1) Find the probability $P(X_n = \max(X_1, \cdots, X_n))$. For the rest, assume that the random variables are iid Gaussian with mean zero and variance one. (2) Find the mgf of X_n. (3) Find the jpdf in the vector-matrix form of the random vector $(X_1, X_2, \cdots, X_n)^t$, where the superscript t denotes the transpose.

2. Let $X_1, X_2, \cdots$ denote iid random variables with the common mgf $\phi(s)$, and let N be a non-negative integer-valued random variable with the pmf $p_N(n)$ that is independent of $X_1, X_2, \cdots$. Let also $R = X_1 + X_2 + \cdots + X_N$ in which the number of terms to be summed is random. (1) Find the mgf of R. For the rest, assume that $X_1, X_2, \cdots$ are iid Gaussian with mean one and variance one, and N is a geometric random variable with $p_N(n) = p(1-p)^{n-1}$, $n = 1, 2, \cdots$. (2) Find the mgf of N. (3) Find the mgf of R, where we consider the sum is zero when $N = 0$. You can use the Gaussian mgf without derivation.

3. The sample space has four equiprobable sample points that are linked to the four sample paths, $\cos t$, $\cos(t - \pi/2)$, $\cos(t - \pi)$, and $\cos(t - 3\pi/2)$, which defines the random process $X(t)$. (1) Find the mean function. (2) Find the acf. (3) State if $X(t)$ is wss, and justify your answer. (4) State if $X(t)$ is stationary (in strict sense), and justify your answer.

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Intro to RVs and RPs, 4th exam, 2008. 5. 27

1. (1) $P(X_n = \max(X_1, \cdots, X_n)) = P(X_1 \le X_n, \cdots, X_{n-1} \le X_n)$

$= \int_{-\infty}^{\infty} P(X_1 \le x, \cdots, X_{n-1} \le x) f(x)\,dx = \int_{-\infty}^{\infty} F^{n-1}(x) f(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx}\left(\frac{F^n(x)}{n}\right) dx$

$= \frac{1}{n} F^n(x) \Big|_{-\infty}^{\infty} = \frac{1}{n}(1-0) = 1/n$. (2) $\phi_{X_n}(s) = Ee^{sX} = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx - x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-s)^2/2} e^{s^2/2} dx = e^{s^2/2}$.

(3) $C_{\underline{X}} = I$, $\mu_{\underline{X}} = \underline{0}$. $f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} |\det C_X|^{1/2}} \exp\left(-\frac{1}{2}(\underline{x} - \mu_{\underline{X}})^t C_{\underline{X}}^{-1} (\underline{x} - \mu_{\underline{X}})\right)$

$= \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} \underline{x}^t \underline{x}\right)$

2. (1) $\phi_R(s) = Ee^{sR} = EE(e^{sR}|N) = \sum_{n=0}^{\infty} E(e^{sR}|N=n)\, p_N(n)$

$= \sum E e^{s(X_1 + \cdots + X_n)} p_N(n) = \sum Ee^{sX_1} \cdots Ee^{sX_n} p_N(n) = \sum \phi^n(s)\, p_N(n)$

$= \sum e^{n \ln \phi(s)} p_N(n) = \phi_N(\ln \phi(s))$. (2) $\phi_N(s) = Ee^{sN}$

$= \sum_{n=1}^{\infty} e^{sn} p(1-p)^{n-1} = \frac{p}{1-p} \sum_{n=1}^{\infty} (e^s(1-p))^n = \frac{p}{1-p} \frac{e^s(1-p)}{1-e^s(1-p)} = \frac{pe^s}{1-(1-p)e^s}$

(3) $\phi(s) = e^{s + s^2/2}$. $\phi_R(s) = \frac{pe^{s+s^2/2}}{1-(1-p)e^{s+s^2/2}}$

3. (1) Note that $\cos(t - \pi/2) = \sin t$, $\cos(t-\pi) = -\cos t$, and $\cos(t - 3\pi/2) = -\sin t$. $\mu_X(t) = EX(t) = \frac{1}{4}(\cos t + \cos(t-\pi/2) + \cos(t-\pi) + \cos(t - 3\pi/2)) = 0$. (2) $R_X(t_1, t_2) = EX(t_1)X(t_2) = \frac{1}{4}(\cos t_1 \cos t_2 + \sin t_1 \sin t_2 + (-\cos t_1)(-\cos t_2) + (-\sin t_1)(-\sin t_2)) = \frac{1}{2}\cos(t_1 - t_2)$. (3) $X(t)$ is wss because $\mu_X(t) = 0$, a constant, and $R_X(t_1, t_2)$ is a function of $t_1 - t_2$ and is shift-invariant. (4). $X(0)$ takes the values -1, 0, and 1 with probabilities $1/4$, $1/2$, and $1/4$, respectively, while $X(\pi/4)$ takes $\sqrt{2}/2$ and $-\sqrt{2}/2$ with equal probability. Therefore it is <u>not</u> stationary in strict sense.

(5) 기말고사

Introduction to Random Variables and Random Processes, Exam 5, Final, 2008.6.5.

Total 200 points: 1:40 (4 each); 2:(1)13, (2)13, (3)13; 3:(1)12, (2)16, (3)13; 4:(1)12, (2)14, (3)15; 4:13; 5:(1)13, (2)13

1. Classify the following into T for true and F for false.
   (a) If X and Y are equal, $F_X(u) = F_X(u)$ for all u.
   (b) If X and Y are identical, $F_X(u) = F_X(u)$ for all u.
   (c) If $F_X(u) = F_X(u)$ for all u, X and Y are equal.
   (d) If $F_X(u) = F_X(u)$ for all u, X and Y are identical.
   (e) If X and Y are independent, their covariance is zero.
   (f) If X and Y are not independent, their covariance is nonzero.
   (g) If the covariance of X and Y is zero, they are independent.
   (h) If the covariance of X and Y is nonzero, they are not independent.
   (i) If two events A and B are independent, $A \cap B = \emptyset$.
   (j) If $A \cap B = \emptyset$, the two events A and B are independent.

2. X and Y are binary random variables with values 0 or 1, and $P(X = 0) = 1/4$. (1) Find the jcdf $F_{X,Y}(x, y)$ when X and Y are equal. (2) Find the jcdf when they are uncorrelated. (3) Find the jcdf when they are orthogonal.

3. The jpdf of random variables X and Y is given as $f_{X,Y}(x,y) = \begin{cases} 3/2, & x \geq 0, y \geq 0, x+y \leq 1 \\ 1/2, & x \leq 1, y \leq 1, x+y \geq 1 \end{cases}$. (1) Find the pdf $f_X(x)$. (2) Find $E(X|Y)$. (3) Find $P(X \leq 3/4 \mid Y \geq 1/4)$.

4. Given the sample space S with three sample points, s_1, s_2, and s_3 and respective probability allocation 1/2, 1/4, and 1/4, let for $n = \cdots, -1, 0, 1, \cdots$,
   $X(n,s) = \begin{cases} 2, & s = s_1 \\ 1 + (-1)^n, & s = s_2 \\ 1 - (-1)^n, & s = s_3 \end{cases}$ (1) Find the cdf $F_{X(0)}(x)$, and plot it. (2) State if $X(t)$ is wss and why. (3) State if $X(t)$ is sss (strict-sense stationary) and why.

5. Customers arrive at a bank as a Poisson process of rate λ per minute. Each arriving customer is immediately served for two minutes and leaves the bank without delay. Find the pmf of the number of customers in the bank.

6. A white noise goes through an ideal bandpass filter with the center frequency f_0 and bandwidth W Hertz ($W \ll f_0$), and the output is called $X(t)$. (1) Find the acf $R_X(\tau)$. (2) Find all the properties that the sequence $X(n/W)$, $n = \cdots, -1, 0, 1, \cdots$ has.

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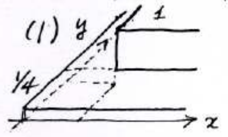
Intro to RVs and RPs, Final, 2008.6.5.

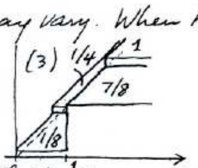
1. (a) T (b) T (c) F (d) T (e) T (f) F (g) F (h) T (i) F (j) F

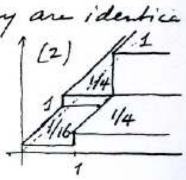
2. (1) $p_{XY}(0,0) = 1/4$, $p_{XY}(1,1) = 3/4$. $F_{XY}(x,y) = 1$ for $x, y \geq 1$, $F_{XY}(x,y) = 1/4$ for $x, y \geq 0$ but not $x, y \geq 1$, and $F_{XY}(x,y) = 0$ elsewhe



(2) Note that $EXY = p_{XY}(1,1)$, $EX = p_X(1)$, and $EY = p_Y(1)$. Uncorrelatedness implies that $EXY = p_{XY}(1,1) = EXEY = p_X(1)\,p_Y(1)$. Let $p_Y(1) = p$, then $p_{XY}(1,1) = 3p/4$.

$p_{XY}(1,0) = p_X(1) - p_{XY}(1,1) = 3(1-p)/4$, $p_{XY}(0,1) = p_Y(1) - p_{XY}(1,1) = p/4$, and $p_{XY}(0,0) = (1-p)/4$. Therefor X & Y are independent. This, however, does not fully specify $p_{XY}(x,y)$, so the answer may vary. When they are identica $F_{XY}(x,y)$ is given by the picture (2).





(3) $EXY = p_{XY}(1,1) = 0$ by orthogonality.

$p_{XY}(1,0) = p_X(1) - p_{XY}(1,1) = 3/4$. If we let $p_Y(1) = p$,

$p_{XY}(0,1) = p_Y(1) - p_{XY}(1,1) = p$ and $p_{XY}(0,0) = p_X(0) - p_{XY}(0,1) = 1/4 - p$.

This also does not fully specify $p_{XY}(x,y)$, so the answer may vary. When $p = 1/8$, $F_{XY}(x,y)$ is given by the picture (3).

3. (1) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_0^{1-x} 3/2\,dy + \int_{1-x}^{1} 1/2\,dy = \frac{3}{2} - x, & 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$

(2) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \longrightarrow = \begin{cases} \frac{3}{2} - y, & 0 \leq y \leq 1 \\ 0, & \text{else.} \end{cases}$ (or simply by symmetry, $f_Y(y) = f_X(y)$)

$f_{X|Y}(x|y) = f_{XY}(x,y)/f_Y(y)$

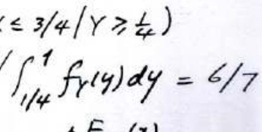
$$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_0^{1-y} \frac{3}{2}\,\frac{x}{\frac{3}{2}-y}\,dx + \int_{1-y}^{1} \frac{1}{2}\,\frac{x}{\frac{3}{2}-y}\,dx$$

$= \frac{3-4y+2y^2}{6-4y}$ for $0 \leq y \leq 1$. It is not defined elsewhere.

Therefore $E(X|Y) = (3-4Y+2Y^2)/(6-4Y)$. (3) $P(X \leq 3/4 \mid Y \geq \frac{1}{4})$

$$= P(X \leq 3/4, Y \geq 1/4)/P(Y \geq 1/4) = \int_{1/4}^{1}\int_0^{3/4} f_{XY}(x,y)\,dx\,dy \Big/ \int_{1/4}^{1} f_Y(y)\,dy = 6/7$$

4. (1) $F_{X(0)}(x) = P(X(0) \leq x) = \begin{cases} 0, & x < 0 \\ P(s_3) = 1/4, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$



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(2) $\mu_X(n) = EX(n) = 2 \cdot 1/2 + (1+(-1)^n) \cdot 1/4 + (1-(-1)^n) \cdot 1/4 = 3/2$

$R_X(m,n) = EX(m)X(n) = 2 \cdot 2 \cdot 1/2 + (1+(-1)^m)(1+(-1)^n)\,1/4 + (1-(-1)^m)(1-(-1)^n)\,1/4$

$= 2 + \frac{1}{4}(1+(-1)^m+(-1)^n+(-1)^{m+n}) + \frac{1}{4}(1-(-1)^m-(-1)^n+(-1)^{m+n})$

$= \frac{5}{2} + \frac{1}{2}(-1)^{m+n} = \frac{5}{2} + \frac{1}{2}(-1)^{m-n}$ because $m+n$ and $m-n$ become even or odd simultaneously. Therefore $X(n)$ is WSS.

(3) Consider. $F_{X(n_1)\cdots X(n_k)}(x_1, \cdots, x_k)$ and $F_{X(n_1+l)\cdots X(n_k+l)}(x_1, \cdots, x_k)$.
Without loss of generality we need only to consider $x_i = 1$ because if $x_i \geq 2$, $X(n_i) \leq x_i$ is a redundant condition, and if $x_i < 0$, the two cdfs are both zero and equal. If $n_1, \cdots, n_k$ are all odd, the left cdf $= P(\{s_3\}) = 1/4$, and $n_1+l, \cdots, n_k+l$ are either all odd (l is even) or even (l is odd) and the right cdf is either $P(\{s_3\}) = 1/4$ or $P(\{s_2\}) = 1/4$, respectively. So the equality holds, and also for $n_1, \cdots, n_k$ all even. If even and odd numbers are mixed among $n_1, \cdots, n_k$, both cdfs must be zero. Therefore it's stationary.

5. It's just the number of arrivals in 2 minuts. Lets call it N. N is a poisson random variable with parameter λt, where $0 \leq t < 2$ within 2 minutes from the opening of the bank, and $t = 2$ afterwards

$$p_N(n) = \begin{cases} \frac{(\lambda t)^n}{n!} e^{-\lambda t} & \text{within 2 min.} \\ \frac{(2\lambda)^n}{n!} e^{-2\lambda} & \text{afterwards.} \end{cases}$$

6. The psd of $X(t)$ is $N_0/2$ times the ideal BP response. (1) $R_X(\tau) =$

$$\int_{-f_0-W/2}^{-f_0+W/2} \frac{N_0}{2} e^{+j2\pi f\tau} df + \int_{f_0-W/2}^{f_0+W/2} \frac{N_0}{2} e^{j2\pi f\tau} df = \frac{N_0}{\pi\tau} \cos 2\pi f_0\tau \sin \pi W\tau$$

(2) $EX(m/W)X(n/W) = R_X((m-n)/W) = \begin{cases} N_0 W, & m=n \\ 0, & m \neq n \end{cases}$. Considering that $X(t)$ is a bandpass signal with $EX(t)=0$,
We can conclude that $X(n/W)$, $n = \cdots -1, 0, 1, \cdots$ are uncorrelated and orthogonal random variables.