Nonlinear Systems Midterm-Exam. (Closed-book, 2005 / 2)

1. Determine whether the following function is locally Lipschitz/globally Lipschitz:
$$f(x) = \begin{bmatrix} x_1 \exp(-x_2^2) \\ x_2 \exp(-x_1^2) \end{bmatrix}.$$

2. Consider the scalar system
$$\dot{x} = kx^n \qquad \text{with } k \text{ a real number and } n \text{ a positive integer}$$
   (a) When is the equilibrium $x = 0$ globally asymptotically stable?
   (b) When is the equilibrium $x = 0$ globally exponentially stable?
   (c) When does the system have finite escape time?

3. How many limit cycles does the following system have? (Zero if there is none.) If the limit cycles exist, determine their stability.
$$\begin{aligned} \dot{x} &= x \sin(x^2 + y^2) - y \\ \dot{y} &= y \sin(x^2 + y^2) + x \end{aligned}$$
(You may want to consider a coordinate change.)

4. For a system
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2, \end{aligned}$$
show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x(t)$ approaches $\bar{x}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \to \infty$.

5. Consider the single-input nonlinear system
$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^n,$$
where f and g are C^1 functions, and $f(0) = 0$. Suppose the uncontrolled system $\dot{x} = f(x)$ is stable and let V be a C^1 positive definite and radially unbounded Lyapunov function satisfying
$$\frac{\partial V}{\partial x} f(x) \leq 0 \qquad {}^{\forall} x \in \mathbb{R}^n.$$
Suppose $\frac{\partial V}{\partial x} g(x) \neq 0$ if $\frac{\partial V}{\partial x} f(x) = 0$ except the origin. Show that, with the feedback control
$$u = -g^T(x) \left(\frac{\partial V}{\partial x} \right)^T,$$
the origin of the closed-loop system is GAS.