Nonlinear Systems Midterm-Exam. (Open-book, 2005 / 2)

1. Determine the stability property of

$$\dot{x} = f(x) = \begin{bmatrix} -6x_1 + 2x_2 \\ 2x_1 - 6x_2 - 2x_2^3 \end{bmatrix}, \qquad x \in \mathbb{R}^2.$$

(You may want to try it with $V(x) = x^T x$, or $V(x) = f(x)^T f(x)$.) Provide your rigorous reasoning.

2. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3(1 - x_1^2) - x_2.$$

(a) Show that $V(x) = \frac{1}{4}x_1^2 - \frac{1}{6}x_1^6 + \frac{1}{2}x_2^2$ is positive definite in some neighborhood of the origin.

(b) Using the above $V(x)$ what can you conclude about the origin among the following?

i. The origin is local stable because __________.

ii. The origin is locally asymptotically stable because __________, and my estimate of the region of attraction is __________.

iii. The origin is unstable because __________.

(c) Show that the origin is not globally asymptotically stable.

3. Suppose that, for a smooth nonlinear system $\dot{x} = f(x)$, there exists a Lyapunov function $V(x)$ such that

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$
$$\frac{\partial V}{\partial x}(x)f(x) \le -c_3\|x\|^2$$
$$\left\| \frac{\partial V}{\partial x}(x) \right\| \le c_4\|x\|$$

for all x, where c_i's are positive constants. Now consider $\dot{x} = f(x) + g(x)$ where $\|g(x)\| \le \delta$ (δ is a positive constant). In this case, we know that any solution $x(t)$ converges to a ball of radius $\rho > 0$. With the above information, determine the smallest value of ρ and explain your approach.

4. Consider a time-varying linear system

$$\dot{x} = A(t)x.$$

Prove or disprove that, if $(A(t) + A^T(t))$ is negative definite for every fixed t, then the origin is GES. (You may want to use $V(x) = \|x\|^2$.) If the origin is GES, then estimate the rate of convergence. If not, provide your condition so that the origin is GES.