Nonlinear Systems Midterm Exam. (Closed-book, 2006 / 2)

1. (5pts) Let α be a class-$\mathcal{K}$ function on $[0, a)$ with $a > 0$. Show that

$$\alpha(x_1 + x_2) \leq \alpha(2x_1) + \alpha(2x_2), \qquad {}^{\forall}x_1, x_2 \in [0, \frac{a}{2}).$$

2. (10pts) The scalar system

$$\dot{x} = -x + x^2 u$$

is not input-to-state stable (ISS). Suggest a feedback $\alpha(x)$ such that the following control

$$u = \alpha(x) + v$$

makes the above system ISS with the new input v. Justify your answer briefly.

3. (10pts) Provide an example system for which the origin is an equilibrium that is not exponentially stable but asymptotically stable. Explain your answer briefly.

4. (15pts) Prove that the equilibrium at $x = 0$ of the system

$$\ddot{x} + \dot{x}^3 + x^5 \sin^2 x = 0$$

is LAS. Estimate your region of attraction. (Hint: Consider $V(x, \dot{x}) = \int_0^x y^5 \sin^2 y dy + \frac{1}{2}\dot{x}^2$.)

5. (10pts) Consider the system

$$\dot{x}_1 = x_2 - g(t)x_1(x_1^2 + x_2^2), \qquad \dot{x}_2 = -x_1 - g(t)x_2(x_1^2 + x_2^2)$$

where $K \geq g(t) \geq k > 0$ for all $t \geq 0$. Is the origin uniformly asymptotically stable? Is it exponentially stable?

6. (15pts) For a system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n$$

where f is a C^1 vector field having the property $f(0) = 0$. Suppose that there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -\alpha_3(\|x\|), \qquad {}^{\forall}x$$

where α_i's are class-$\mathcal{K}_\infty$ functions. Let $B_r = \{x : \|x\| \leq r\}$ where $r > 0$ is given. Prove that the solution from any initial condition $x_0 \in \mathbb{R}^n$ such that $\|x_0\| > r$, enters the set B_r in finite-time. If the elapsed time is denoted by T (depending on x_0), can you get the upper-bound of T using those α_i functions?