#1. Let $\tau = \omega t$.

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\omega}\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \varepsilon \begin{bmatrix} x_2 \\ -x_2 - \sin(\tau)\left(x_1 + \sin(\tau)\right)^2 \end{bmatrix} \triangleq \varepsilon F(x,\tau)$$

(by $\varepsilon = \frac{1}{\omega} \ll 1$)

standard form for averaging theory.

$$F_{av}(x) = \frac{1}{2\pi}\int_0^{2\pi} F(x,s)\,ds = \begin{bmatrix} x_2 \\ -x_2 - \boxed{} \end{bmatrix}$$

$$\sin(\tau)\left[x_1^2 + 2x_1\sin(\tau) + \sin^2(\tau)\right]$$

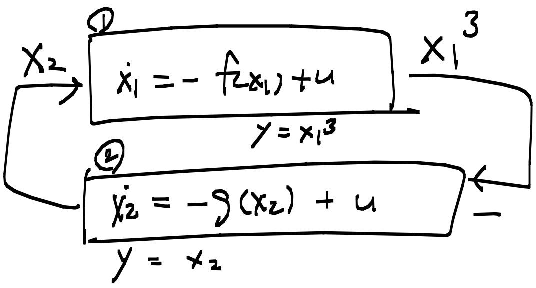
$$= \sin(\tau)\,x_1^2 + 2x_1\sin^2(\tau) + \sin^3(\tau)$$

(average $= 0$, $\underbrace{2x_1\frac{1-\cos(2\tau)}{2}}_{x_1}$, 0)

$$\therefore F_{av}(x) = \begin{bmatrix} x_2 \\ -x_2 - x_1 \end{bmatrix}$$

$\therefore$ origin is spiral. AS. for $\varepsilon \ll 1$.

#2.



$\Rightarrow$ ① is strictly pas. ($\because$) $V_1(x_1) = \frac{1}{4} x_1^4$.

$$\dot{V}_1 = -x_1^3 f(x_1) + \underbrace{x_1^3}_{= y} \cdot \underbrace{u}_{= u}$$

② is strictly pas. ($\because$) $V_2(x_2) = \frac{1}{2} x_2^2$

$$\dot{V}_2 = -x_2 g(x_2) + \underbrace{x_2}_{= y} \underbrace{u}_{= u}$$

Negative feedback.

$\therefore$ GAS.

#3. HJB - ineq. : $L_f V + \frac{1}{2\gamma^2}(L_g V)^2 + \frac{1}{2} h^2 \leq 0$

$\dot{x}_1 = -x_1^3 + x_2 \Rightarrow \dot{x}_1 = -x_1^3 + u$

$y = x_1^3$

$\Rightarrow \nabla V(-x_1^3) + \frac{1}{2\gamma^2}(\nabla V)^2 + \frac{1}{2} x_1^6 = 0$

$\Rightarrow (\nabla V)^2 - 2\gamma^2 x_1^3 (\nabla V) + \gamma^2 x_1^6 = 0$

$(\nabla V) = \gamma^2 x_1^3 \pm \sqrt{\gamma^4 x_1^6 - \gamma^2 x_1^6}$

$\Rightarrow (\gamma^2 \pm \gamma\sqrt{\gamma^2 - 1})\, x_1^3$

let $\gamma_1 = 1$. $(\nabla V) = x_1^3$

$\therefore \underline{V = \frac{1}{4} x_1^4}$

$\dot{x}_2 = -x_2 + \frac{1}{2} x_1^3 \Rightarrow \dot{x}_2 = -x_2 - \frac{1}{2} u$

$y = x_2$

$\Rightarrow (\nabla W)(-x_2) + \frac{1}{2\gamma^2}(\nabla W \frac{-1}{2})^2 + \frac{1}{2} x_2^2 = 0$

$\Rightarrow (\nabla W)^2 - 8\gamma^2 x_2 (\nabla W) + 4\gamma^2 x_2^2 = 0$

$\nabla W = 4\gamma^2 x_2 \pm \sqrt{16\gamma^4 x_2^2 - 4\gamma^2 x_2^2}$ - Let $\gamma_2 = \frac{1}{2}$

$= 4\gamma^2 x_2 = x_2$ $\therefore \underline{W = \frac{1}{2} x_2^2}$

$\therefore \gamma_1 \gamma_2 = \frac{1}{2} < 1$.

By small-gain THM, the origin is GAS.

#4. Center manifold : $v = \pi(w) = k_1 w^2 + k_2 w^3 + O(w^4)$

$$\dot{w}\big|_{v=\pi(w)} = \left(k_1 w^3 + k_2 w^4 + O(w^5)\right) + \left(a w^3\right) + bw\left(k_1 w^4 + k_1 k_2 w^5 + O(w^5)\right)$$

$$\frac{\partial \pi}{\partial w} \cdot \dot{w} = \left(2k_1 w + 3k_2 w^2 + O(w^3)\right)\left(\quad " \quad \right)$$

$$= \dot{v}\big|_{v=\pi(w)} = -\left(k_1 w^2 + k_2 w^3 + O(w^4)\right) + c w^2$$

coefficient for w^2 : $c - k_1 = 0$

" w^3 : $-k_2 = 0$

" w^4 : ?

$\boxed{\therefore k_1 = c \quad \therefore k_2 = 0.}$

Reduced system : $\dot{w}\big|_{v=\pi(w)} = (c+a) w^3 + O(w^4)$

$\therefore \boxed{c + a < 0}$ for stability.

$\boxed{b : \text{any}}$.

#5.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_3$$
$$\underbrace{\frac{1}{k}}_{=\varepsilon} \dot{x}_3 = -x_3 + \sin x_1 + x_2 . \quad \& \quad A.S.$$

$$x_3 = \sin x_1 + x_2$$

Reduced sys. : $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 - x_2 \end{cases}$ ~~A~~

$$V = \underbrace{\int_0^{x_1} \sin(s)\, ds}_{} + \frac{1}{2} x_2^2 = 1 - \cos(x_1) + \frac{1}{2} x_2^2$$

$$\dot{V} = \cancel{\sin(x_1) x_2} - \cancel{\sin(x_1) x_2} - x_2^2$$

By LaSalle. $\underline{LAS}$

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#6. By THM 9.1,

if $Z(s) = \dfrac{1}{1-\alpha\dfrac{1}{s^2+2s+1}} = \dfrac{s^2+2s+1}{s^2+2s+(1-\alpha)}$ is SPR

then Abs. Stable

① $1-\alpha > 0$ for Hurwitz

② $2\,\mathrm{Re}\,Z(j\omega) = \dfrac{1-\omega^2+2j\omega}{1-\alpha-\omega^2+2j\omega} + \dfrac{(1-\omega^2)-2j\omega}{(1-\alpha-\omega^2)-2j\omega}$

$$= \frac{\left[(1-\omega^2)(1-\alpha-\omega^2)+4\omega^2+j\left(\underbrace{2\omega(1-\alpha-\omega^2)-2\omega(1-\omega^2)}_{=\,-\alpha\cdot 2\omega(1-\omega^2)}\right)\right] + [\,''\,]^*}{(1-\alpha-\omega^2)+4\omega^2}$$

$$[\,''\,] = (1-\omega^2)^2-\alpha(1-\omega^2)+4\omega^2 - j\,\alpha\,2\omega(1-\omega^2)$$

$$= \underline{1-2\omega^2+\omega^4-\alpha+\alpha\omega^2+4\omega^2} - \quad ''$$

$$= \omega^4+(2+\alpha)\omega^2+(1-\alpha)$$

$$\forall\ \mathrm{Re}\,Z(j\omega) = \frac{\omega^4+(2+\alpha)\omega^2+(1-\alpha)}{(1-\alpha)+3\omega^2}$$

∀ω 양수

∀ω 양수

O.K

$$\boxed{\therefore\ \ \alpha = 1}$$