

Eng Math2. Mid Term (10/29/2008)

(Closed book and note: 90 min. Total 200)

1. Evaluate the following integral (you need to give proper reasoning):

$$\int_0^{\infty} \frac{\sin w}{w} dw$$

Sol)

When you find the Fourier integral of the following function,

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ 1 & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

Period is infinite.

$$\int_0^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2}$$

2. It has been well known that surface integration of electric field over a closed surface results in q/ϵ_0 (Gaussian law) where q is point charge enclosed by the

surface, and ϵ_0 is the permittivity of free space: $\oint_S \vec{E} \cdot \vec{n} da = \frac{q}{\epsilon_0}$. It is also well

known that \vec{F} (force) = $q\vec{E}$ and φ (electrostatic potential) = $-\int_{\vec{r}_{ref}}^{\vec{r}} \vec{E} \cdot d\vec{r}$.

Evaluate the followings:

(a) $\oint \nabla \varphi \cdot d\vec{r} = ?$

(b) $\nabla \cdot \vec{E} = ?$

(c) $\nabla \times \vec{E} = ?$

Sol)

(a) since the electrostatic potential is defined as the negative gradient of potential, the integration is path-independent. $\oint \nabla \varphi \cdot d\vec{r} = 0$

(b) from the divergence theorem: $\nabla \cdot \vec{E} = \frac{q}{\epsilon_0}$.

(c) as in (a), $\nabla \times \vec{E} = \vec{0}$.

3. Answer to each question properly:

(a) simplify $\frac{d}{dt} [\vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}''(t))]$.

Sol)

$$\begin{aligned} \frac{d}{dt} [\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] &= \mathbf{r}(t) \cdot \frac{d}{dt} (\mathbf{r}'(t) \times \mathbf{r}''(t)) + \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) \\ &= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t) + \mathbf{r}''(t) \times \mathbf{r}''(t)) + \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) \\ &= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t)) \end{aligned}$$

(b) Evaluate the integral $\oint_C (zdx + xdy + ydz)$ where C is the trace of the cylinder $x^2+y^2=1$ in the plane $y+z=2$. Orient C counterclockwise as viewed from above.

Sol) Let the \mathbf{F} is $zi+xj+yk$. So, the curl of \mathbf{F} is $\mathbf{i}+\mathbf{j}+\mathbf{k}$. Let $g(x,y,z)=y+z-2=0$. Then,

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{1}{\sqrt{2}} [0, 1, 1]. \text{ From the Stoke's thm,}$$

$$\oint_C (zdx + xdy + ydz) = \int_S \vec{F} \cdot \vec{n} dA = \sqrt{2} \int_S dA = \sqrt{2} \sqrt{2} \pi = 2\pi$$

4. Answer to each question properly:

(a) Find the Fourier transform of $\frac{d}{dx}[f(x) * g(x)]$.

Sol) $iF(q)G(q)$

(b) Find the inverse Fourier transform of 1.

Sol) $\delta(x)$

5.

(solution)

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos \omega x dx \quad - \textcircled{a}$$

$$\begin{aligned} \int_0^{\infty} \frac{e^{-ax} \cos \omega x dx}{\frac{d}{dx} \frac{u}{u}} &= -\frac{1}{a} e^{-ax} \cos \omega x \Big|_0^{\infty} + \omega \int_0^{\infty} -\frac{1}{a} e^{-ax} \sin \omega x dx \\ &= \frac{1}{a} - \frac{\omega}{a} \int_0^{\infty} e^{-ax} \sin \omega x dx \quad - \textcircled{1} \end{aligned}$$

그런데

$$\begin{aligned} \int_0^{\infty} \frac{e^{-ax} \sin \omega x dx}{\frac{d}{dx} \frac{u}{u}} &= -\frac{1}{a} e^{-ax} \sin \omega x \Big|_0^{\infty} - \omega \int_0^{\infty} -\frac{1}{a} e^{-ax} \cos \omega x dx \\ &= 0 + \frac{\omega}{a} \int_0^{\infty} e^{-ax} \cos \omega x dx \quad - \textcircled{2} \end{aligned}$$

②를 ①에 대입하면

$$\int_0^{\infty} e^{-ax} \cos \omega x dx = \frac{1}{a} - \frac{\omega^2}{a^2} \int_0^{\infty} e^{-ax} \cos \omega x dx$$

$$\left[1 + \frac{\omega^2}{a^2}\right] \int_0^{\infty} e^{-ax} \cos \omega x dx = \frac{1}{a}$$

$$\frac{a^2 + \omega^2}{a^2} \int_0^{\infty} e^{-ax} \cos \omega x dx = \frac{1}{a}$$

$$\int_0^{\infty} e^{-ax} \cos \omega x dx = \frac{a}{a^2 + \omega^2} \quad - \textcircled{3}$$

이것을 ③에 대입하면

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right)$$

6. (a)

$$u(x) * g(x) = ?,$$

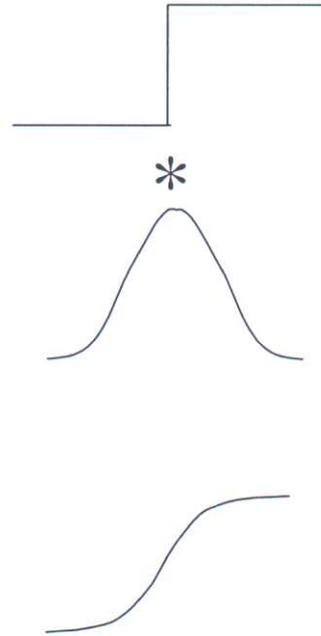
$u(x)$: 1 when $x \geq 0$, 0 when $x < 0$.

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$u(x) * g(x)$$

$$= \int_{-\infty}^{\infty} u(p)g(x-p)dp$$

$$= \int_0^{\infty} g(x-p)dp = \int_{-\infty}^x g(\tau)d\tau \quad (\text{replacing } x-p=\tau)$$



(b)

$$\hat{f}[e^{-x^2}] = \frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}$$

$$\text{Reminding } \hat{f}[f'(x)] = iw\hat{f}[f(x)], \hat{f}[-2xe^{-x^2}] = iw\frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}.$$

$$\text{since } \hat{f}[f(x-a)] = e^{-iwa} \hat{f}[f(x)],$$

$$\hat{f}[(x-a)e^{-(x-a)^2}] = -iw\frac{1}{2\sqrt{2}} e^{-\frac{w^2}{4}-iwa}$$

Engineering Math Midterm

(October 29, 2004)

(closed book & note : 90min)

Answer Sheet

7. $\underline{r}(t) = (a \cos t, a \sin t, ct)$

(1) direction

from the figure, $c > 0 \rightarrow$ "counterclockwise" \leftarrow Ans.
(if $c < 0 \rightarrow$ "clockwise" direction).

(2) tangential vector

$$\underline{r}'(t) = \frac{d\underline{r}(t)}{dt} = (-a \sin t, a \cos t, c) \leftarrow \text{Ans.}$$

(3) length of the curve (circular helix)

$$\begin{aligned} L &= \int_0^t \sqrt{\underline{r}' \cdot \underline{r}'} dt = \int_0^t \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + c^2} dt \\ &= \int_0^t \sqrt{a^2 + c^2} dt = t \sqrt{a^2 + c^2} \leftarrow \text{Ans.} \end{aligned}$$

(for one full turn: $L_{t=0 \rightarrow 2\pi} = 2\pi \sqrt{a^2 + c^2}$)

8. $r = \sqrt{x^2 + y^2 + z^2}$ in axisymmetric spherical coordinate.
(indep. of θ & ϕ)

(a) $\nabla \phi(r)$ by chain rule

$$\nabla \phi(r) = \frac{\partial \phi}{\partial r} \nabla r$$

$$\nabla r = \frac{\partial r}{\partial x} \hat{i} + \frac{\partial r}{\partial y} \hat{j} + \frac{\partial r}{\partial z} \hat{k}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

likewise, $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

(2)

$$\begin{aligned} \therefore \nabla \phi(r) &= \left(\frac{\partial \phi}{\partial r} \right) \frac{1}{r} \underbrace{(x\hat{i} + y\hat{j} + z\hat{k})}_r \\ &= \boxed{\frac{r}{r} \left(\frac{\partial \phi}{\partial r} \right)} \leftarrow \text{Ans.} \end{aligned}$$

$$\text{c) } \nabla^2 \phi(r) = (\nabla \cdot \nabla) \phi(r)$$

$$\begin{aligned} \nabla \cdot \left(\frac{r}{r} \frac{\partial \phi}{\partial r} \right) &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{x}{r} \frac{\partial \phi}{\partial r} \hat{i} + \frac{y}{r} \frac{\partial \phi}{\partial r} \hat{j} + \frac{z}{r} \frac{\partial \phi}{\partial r} \hat{k} \right) \\ &= \frac{\partial}{\partial x} \left[\frac{x}{r} \frac{\partial \phi}{\partial r} \right] + \frac{\partial}{\partial y} \left[\frac{y}{r} \frac{\partial \phi}{\partial r} \right] + \frac{\partial}{\partial z} \left[\frac{z}{r} \frac{\partial \phi}{\partial r} \right] \end{aligned}$$

$$\frac{\partial}{\partial x} \left[\frac{x}{r} \frac{\partial \phi}{\partial r} \right] = \frac{r \frac{\partial}{\partial x} \left[\frac{x}{r} \frac{\partial \phi}{\partial r} \right] - x \frac{\partial \phi}{\partial r} \left(\frac{\partial r}{\partial x} \right)}{r^2} \leftarrow x/r$$

$$= \frac{1}{r} \frac{\partial}{\partial x} \left[x \frac{\partial \phi}{\partial r} \right] - \frac{x^2}{r^3} \frac{\partial \phi}{\partial r}$$

$$= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{x}{r} \frac{\partial^2 \phi}{\partial x \partial r} - \frac{x^2}{r^3} \frac{\partial \phi}{\partial r}$$

$$= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{x}{r} \frac{\partial \phi}{\partial r} \left(\frac{1}{r} \right) - \frac{x^2}{r^3} \frac{\partial \phi}{\partial r}$$

$$= \boxed{\left(\frac{1}{r} - \frac{x^2}{r^3} \right) \frac{\partial \phi}{\partial r} + \frac{x}{r^2} \frac{\partial^2 \phi}{\partial x \partial r}}$$

$$\text{Likewise, } \frac{\partial}{\partial y} \left[\frac{y}{r} \frac{\partial \phi}{\partial r} \right] = \left(\frac{1}{r} - \frac{y^2}{r^3} \right) \frac{\partial \phi}{\partial r} + \frac{y}{r^2} \frac{\partial^2 \phi}{\partial y \partial r}$$

$$\frac{\partial}{\partial z} \left[\frac{z}{r} \frac{\partial \phi}{\partial r} \right] = \left(\frac{1}{r} - \frac{z^2}{r^3} \right) \frac{\partial \phi}{\partial r} + \frac{z}{r^2} \frac{\partial^2 \phi}{\partial z \partial r}$$

$$\therefore \nabla^2 \phi(r) = \left(\frac{3}{r} - \frac{r^2}{r^3} \right) \frac{\partial \phi}{\partial r} + \frac{r^2}{r^2} \frac{\partial^2 \phi}{\partial r^2}$$

$$= \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} = \boxed{\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right)}$$

$$(c) \nabla^2(1/r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \right]$$

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \right) = \frac{-1}{r^2}$$

$$\therefore \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \cdot \left(\frac{-1}{r^2} \right) \right] = \boxed{0} \leftarrow \text{Ans.}$$

$$9. \quad \underline{h} = \left(-\frac{y}{x^2+y^2} \right) \hat{i} + \left(\frac{x}{x^2+y^2} \right) \hat{j}$$

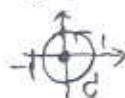
$$(a) \quad \nabla \times \underline{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{k} \left[\frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} \right) \right]$$

$$= \hat{k} \left[\frac{x^2+y^2 - 2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2} \right]$$

$$= \hat{k} \left[\frac{-x^2+y^2 + x^2 - y^2}{(x^2+y^2)^2} \right] = \boxed{0} \leftarrow \text{Ans. (Null vector)}$$

$$(b) \quad \oint_C \underline{h} \cdot d\underline{r} = \oint_C \left(\frac{-y\hat{i} + x\hat{j}}{x^2+y^2} \right) \cdot (dx\hat{i} + dy\hat{j})$$



$$= \oint_C \frac{-ydx + xdy}{x^2+y^2}$$

$$\underline{r} = \cos t \hat{i} + \sin t \hat{j} \quad \left(\begin{array}{l} x \\ y \end{array} \right) \\ \text{C parametric representation}$$

$$= \int_0^{2\pi} \frac{-\sin t (-\sin t) dt + \cos t \cos t dt}{\cos^2 t + \sin^2 t} dt$$

$$= \int_0^{2\pi} dt = \boxed{2\pi} \leftarrow \text{Ans. } (\because \oint_C \underline{h} \cdot d\underline{r} \neq 0)$$

10. (a) $\mathcal{F}[f(x-a)]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{-i\omega x} dx$$

Let $x-a = x'$ $dx = dx'$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-i\omega(x'+a)} dx'$$

$$= e^{-i\omega a} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-i\omega x'} dx'}_{\mathcal{F}[f(x)]}$$

$$= \boxed{e^{-i\omega a} \mathcal{F}[f(x)]} \leftarrow \text{Ans}$$

otherwise,

(b) $\mathcal{F}\left[\int_0^x f(u) du\right]$

$$\int_0^x f(u) du = h(x)$$

$$\frac{\partial h(x)}{\partial x} = f(x)$$

$$\boxed{|x| \rightarrow \infty \quad f(x) \rightarrow 0.}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_0^x f(u) du \right] e^{-i\omega x} dx$$

IBP $\frac{1}{\sqrt{2\pi}(-i\omega)} \int_0^x f(u) du \Big|_{-\infty}^{\infty} - \left(\frac{1}{-i\omega}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

From $\mathcal{F}[h'(x)] = i\omega \mathcal{F}[h(x)]$
 $\mathcal{F}[h(x)] = \frac{\mathcal{F}[h'(x)]}{i\omega}$

$$= \boxed{\frac{1}{i\omega} \mathcal{F}[f(x)]} \leftarrow \text{Ans}$$