

# Eng Math. Final Term (12/17/2008)

(Closed book and note: 120 min.)

1. [20 pts] Answer properly to each question:

(a) Evaluate  $\cos\left(\frac{\pi}{2} + i \ln 2\right)$  [5 pts]

Sol)  $\cos\left(\frac{\pi}{2} + i \ln 2\right) = -\frac{3}{4}i$

(b) Find all values of  $\sin^{-1} \sqrt{5}$  [15 pts]

Sol) From  $\sin w = z = \frac{e^{iw} - e^{-iw}}{2i}$

$$e^{2iw} - 2ize^{iw} - 1 = 0$$

$$e^{iw} = iz + (1 - z^2)^{1/2}$$

$$\sin^{-1} z = -i \ln[iz + (1 - z^2)^{1/2}]$$

So,

$$\sin^{-1} \sqrt{5} = -i \ln[\sqrt{5}i + \sqrt{(1-5)}]$$

$$= \frac{\pi}{2} - i \ln(\sqrt{5} + 2)$$

2.. Find the suitable solution of each question: [30 pts]

(a) Evaluate the following integral  $P.V. \int_0^\infty \frac{1 - \cos x}{x^2} dx$ . [20 pts]

Sol)

The integrand of  $\oint_C \frac{1-e^{iz}}{z^2} dz$  has a simple pole at  $z = 0$ .

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2} dx - \pi i \operatorname{Res}(f(z), 0) = 0.$$

Thus,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1-\cos x - i \sin x}{x^2} dx = \pi.$$

Equating real parts gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx = \pi.$$

Finally,

$$\int_0^{\infty} \frac{1-\cos x}{x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}.$$

- (b) Find the condition for analyticity of  $f(z)$  in cylindrical coordinate. You should describe the detail. [10 pts]

Sol) from Cauchy-Riemann equations,

$$u_r = \frac{v_\theta}{r}$$

$$u_x = v_y, u_r r_x = v_\theta \theta_y$$

$$r_x = x/r$$

$$\theta_y = x/r^2$$

$$\text{therefore, } u_r = \frac{v_\theta}{r}$$

$$v_r = -\frac{u_\theta}{r}$$

$$u_y = -v_x, u_r r_y = -v_\theta \theta_x$$

$$r_y = y/r$$

$$\theta_x = y/r^2$$

$$\text{therefore, } v_r = -\frac{u_\theta}{r}$$

3. [20 pts] Evaluate  $\int_0^\infty e^{-ax^2} \cos bx dx$ . [20 pts]

$$\begin{aligned} & \text{Let } z = x + \frac{bi}{2a} \quad \& \quad \text{from } \oint_C e^{-az^2} \cdot e^{izx} dz \\ & \int_{-\infty}^{\infty} e^{-ax^2} \cdot e^{izbx} dx = - \int_{-\infty}^{\infty} e^{-az^2 \left(x + \frac{bi}{2a}\right)^2} \cdot e^{iz\left(x + \frac{bi}{2a}\right)} dx \\ & \qquad \qquad \qquad = \int_{-\infty}^{\infty} e^{-az^2} \cdot e^{-b^2/4a^2} dx \\ & \int_{-\infty}^{\infty} e^{-az^2} (\cos bx + i \sin bx) dx = 2^{-b^2/4a^2} \int_{-\infty}^{\infty} e^{-az^2} dx \end{aligned}$$

from Gaussian distribution

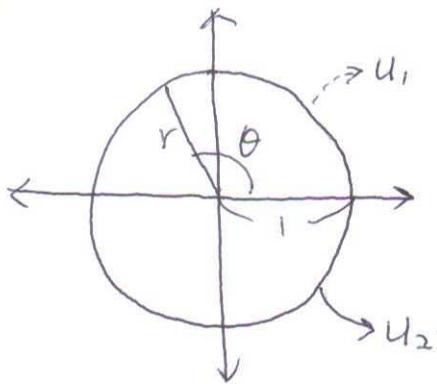
$$\int_{-\infty}^{\infty} e^{-az^2} dx = \frac{\sqrt{\pi}}{a}$$

$$\text{So } \int_0^\infty e^{-ax^2} \cos bx dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/4a^2}$$

4. [30 pts] A circular plate of unit radius whose faces are insulated has upper half of its boundary kept at constant temperature  $u_1$  and the other half at constant temperature  $u_2$ . Find the steady-state temperature of the plate. (Hint: diffusion eqn  $\frac{\partial u}{\partial t} = c^2 \nabla^2 u$ , where  $u$  is the temperature,  $c^2$  is the thermal diffusivity. For cylindrical coordinate,

$$\nabla^2 = \frac{(r^2 u_r)_r}{r^2} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$$

4.



$$U_t = C^2 \left( U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} \right)$$

$= 0$  (for steady state app.)

B.C.  $U(1, \theta) = \begin{cases} U_1, & 0 < \theta < \pi \\ U_2, & \pi < \theta < 2\pi \end{cases}$

$$U(r, \theta) = P(r) \cdot \Phi(\theta)$$

$$P'' \cdot \Phi + \frac{1}{r} \cdot P' \cdot \Phi + \frac{1}{r^2} \cdot P \cdot \Phi'' = 0$$

dividing by  $P \cdot \Phi$

$$\frac{P''}{P} + \frac{1}{r} \cdot \frac{P'}{P} + \frac{1}{r^2} \cdot \frac{\Phi''}{\Phi} = 0$$

$$-r^2 \cdot \frac{P''}{P} - r \cdot \frac{P'}{P} = \frac{\Phi''}{\Phi} \quad \begin{cases} + \rightarrow \text{no soln} \\ 0 \rightarrow \dots \\ - \rightarrow -q^2 < 0 \end{cases}$$

$$\textcircled{1} \quad r^2 P'' + r P' - P q^2 = 0$$

$$P = A_2 \cdot r^q + \frac{B_2}{r^q}$$

$B_2 = 0$  (since T exists when  $r=0$ )

$$\textcircled{2} \quad \Phi'' + \Phi q^2 = 0$$

$$\Phi = A_1 \cos q\theta + B_1 \sin q\theta$$

$$\therefore U(r, \theta) = P \cdot \Phi = A_2 \cdot r^q \cdot [A_1 \cos q\theta + B_1 \sin q\theta]$$

general soln  $U(r, \theta) = \sum_{q=1}^{\infty} r^q (A_q \cos q\theta + B_q \sin q\theta) + A_0 / 2$

~~Final ans~~

$$u(1, \phi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi)$$

From the theory of Fourier Series,

$$A_m = \frac{1}{\pi} \int_0^{2\pi} u(1, \phi) \cdot \cos m\phi d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi} u_1 \cos m\phi d\phi + \frac{1}{\pi} \int_{\pi}^{2\pi} u_2 \cos m\phi d\phi = \begin{cases} 0, & m > 0 \\ u_1 + u_2, & m = 0 \end{cases}$$

$$B_m = \frac{1}{\pi} \int_0^{2\pi} u(1, \phi) \cdot \sin m\phi d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi} u_1 \sin m\phi d\phi + \frac{1}{\pi} \int_{\pi}^{2\pi} u_2 \sin m\phi d\phi$$

$$= \frac{(u_1 - u_2)}{m\pi} \cdot (1 - \cos m\pi)$$

Then,

$$\begin{aligned} u(r, \phi) &= \frac{u_1 + u_2}{2} + \sum_{m=1}^{\infty} \frac{(u_1 - u_2) \cdot (1 - \cos m\pi)}{m\pi} \cdot r^m \sin m\phi \\ &= \frac{u_1 + u_2}{2} + \frac{2(u_1 - u_2)}{\pi} \left[ r \sin \phi + \frac{r^3}{3} \sin 3\phi + \frac{r^5}{5} \sin 5\phi + \dots \right] \end{aligned}$$

Final soln.

(1)

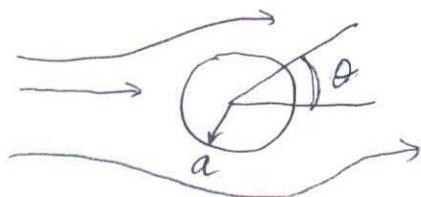
# Engineering Math Trial

(December 17, 2008)

(Closed book & notes: 120min)

## Answer sheet

(Prob1)



$$\underline{U} = \nabla \phi \text{ satisfying } \nabla^2 \phi = 0 \quad \phi(r, \theta)$$

From class with spherical coordinate satisfying the Laplace eqn:

$$\phi(r, \theta) = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta)$$

growing harmonics      decaying harmonics

From BCs:

$$\frac{\partial \phi}{\partial r} = \sum_{n=0}^{\infty} \left( n A_n r^{n-1} - \frac{(n+1) B_n}{r^{n+2}} \right) P_n(\cos \theta)$$

when  $r \rightarrow \infty$  finite velocity ( $V_0 \cos \theta$ ).

$$\therefore A_2 = A_3 = A_4 = \dots = 0$$

$$\frac{\partial \phi}{\partial r}(a, \theta) = \sum_{n=0}^{\infty} \left( n A_n a^{n-1} - \frac{(n+1) B_n}{a^{n+2}} \right) P_n(\cos \theta) = 0$$

$$\therefore B_n = \frac{n}{n+1} a^{2n+1} A_n$$

$$\begin{aligned} \therefore \phi(r, \theta) &= A_0 P_0(\cos \theta) + A_1 \left( r + \frac{a^3}{2r^2} \right) P_1(\cos \theta) \\ &= A_0 + A_1 \left( r + \frac{a^3}{2r^2} \right) \cos \theta. \end{aligned}$$

$$r \rightarrow \infty \quad \frac{\partial \phi}{\partial r} = A_1 \cos \theta = V_0 \cos \theta \quad \therefore A_1 = V_0$$

$$\therefore \boxed{\phi(r, \theta) = \phi_0 + V_0 \left( r + \frac{a^3}{2r^2} \right) \cos \theta} \quad \text{Ans.}$$

(Prob 2)

$$\textcircled{1} \quad f(z) = \frac{e^z - 1}{z^2} = \frac{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}{z^2} = \boxed{\frac{1}{z^2} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots} \quad \text{simple pole!!}$$

$$\textcircled{2} \quad g(z) = \frac{\sin z}{z^2} = \frac{1}{z^2} \left[ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right] = \boxed{\frac{1}{z^2} - \frac{z}{3!} + \frac{z^3}{5!} - \dots} \quad \text{simple pole!!} \quad \text{Ans.}$$

(NOT poles of 2nd order !!)

(Prob 3)

$$f(z) = \frac{1}{1-z^2} = \frac{-1}{(z+1)(z-1)} \quad \text{with center } z=1$$

$$\frac{1}{z+1} = \frac{1}{(z-1)+2} \quad \text{center}$$

$$\textcircled{1} \quad \frac{1}{(z-1)\left\{1 + \frac{z}{z-1}\right\}} \quad \text{if } \left|\frac{z}{z-1}\right| < 1$$

$$= \boxed{\frac{1}{z-1} \sum_{n=0}^{\infty} \left(-\frac{z}{z-1}\right)^n}$$

$$\textcircled{2} \quad \frac{1}{2\left(1 + \frac{z-1}{2}\right)} \quad \text{if } \left|\frac{z-1}{2}\right| < 1$$

$$= \boxed{\frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-1}{2}\right)^n}$$

(3)

$$\frac{1}{1-z^2} = \left[ \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^n}{(2-1)^{n+2}} \quad \text{when } |2-1| > 2 \right] \text{Ans.}$$

$$= \left[ \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2-1)^{n-1}}{2^{n+1}} \quad \text{when } 0 < |2-1| < 2 \right]$$

(Prob 4)

$$(1) \int_0^{2\pi} \frac{d\theta}{1+\varepsilon \cos \theta} \quad (\mid \varepsilon \mid < 1).$$

$$\oint_C \frac{dz/iz}{1+\frac{\varepsilon i}{2}(z+1/z)} = \oint_C \frac{dz}{iz[1+\frac{\varepsilon}{2}z + \frac{\varepsilon}{2}1/z]}$$

$$= \oint_C \frac{dz}{iz[z^2 + \frac{2}{\varepsilon}z + 1]}$$

$$= \frac{2}{\varepsilon i} \oint_C \frac{dz}{z^2 + \frac{2}{\varepsilon}z + 1}$$

$$z^2 + \frac{2}{\varepsilon}z + 1 = 0 \quad z = -\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - 1}$$

$$\begin{cases} z_1 = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2} \\ z_2 = -\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2} \end{cases}$$

$$\text{Res}_{z \rightarrow z_1} \frac{1}{(z^2 + \frac{2}{\varepsilon}z + 1)} = \text{Res}_{z \rightarrow z_1} \frac{1}{2z + 2/\varepsilon} \quad \begin{cases} z_2 \text{ lie outside of the unit circle} \\ \text{while } z_1 \text{ lie inside the unit circle} \end{cases}$$

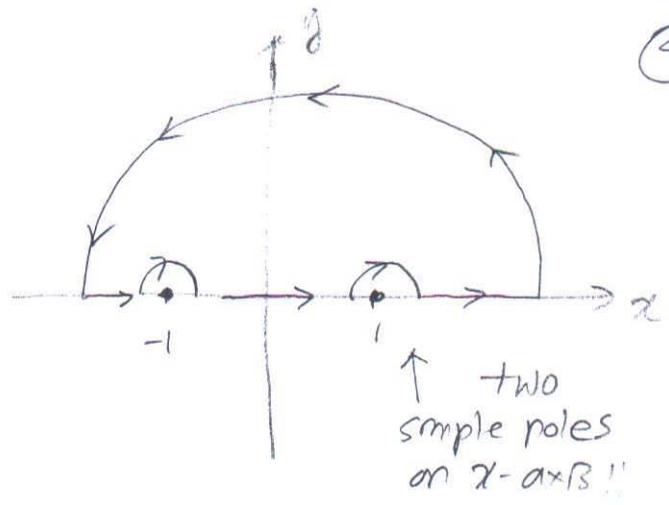
$$= \frac{1}{-\frac{2}{\varepsilon} + \frac{2}{\varepsilon}\sqrt{1-\varepsilon^2} + \frac{2}{\varepsilon}} = \frac{\varepsilon}{2\sqrt{1-\varepsilon^2}}$$

$$\therefore \text{Ans} = 2\pi i \cdot \frac{2}{2\sqrt{1-\varepsilon^2}} = \boxed{\frac{2\pi}{\sqrt{1-\varepsilon^2}}} \leftarrow \text{Ans.}$$

(4)

$$(2) \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi z}{2}\right)}{1-z^2} dz$$

$$= \oint_C \frac{e^{i\frac{\pi z}{2}}}{1-z^2} dz.$$



$$\text{Res}_{z \rightarrow 1} \frac{e^{i\frac{\pi z}{2}}}{1-z^2} = \left. \frac{e^{i\frac{\pi z}{2}}}{-2z} \right|_{z=1} = -\frac{i}{2}$$

$$\text{Res}_{z \rightarrow -1} \frac{e^{i\frac{\pi z}{2}}}{1-z^2} = \left. \frac{e^{i\frac{\pi z}{2}}}{-2z} \right|_{z=-1} = \frac{-i}{2}$$

$$\therefore \text{Ans} = \pi i \left[ -\frac{i}{2} - \frac{i}{2} \right] = \boxed{\pi} \leftarrow \text{Ans.}$$