School of Mech & Aero Eng
Seoul National University

Eng Math 2
Fall '08

MIDTERM

- Do not open exam until told to do so.
- 'How you arrived at your answer' is much more important than the answer itself. Read the following problems carefully, and make sure you show your work *step by step*.
- Please write down the answer *in the given space*. If you need extra space, please use a separate sheet for each problem.
- You may use the *information given in the last page* of this exam.
- Ask questions if you don't understand what the problem says, and **GOOD LUCK !**

Student ID: ________________

Name: ________________

1	/ 18
2	/ 12
3	/ 30
4	/ 20
5	/ 20
Total	/ 100

1. [`3×6=18 pts`] True or False? Explain why, or find the counter example.

(a) If $rank(A) = rank(A^T)$ then A is a square matrix.

(b) If A and B are similar and invertible, then A^{-1} and B^{-1} are also similar.

(c) $det(A + B) = det(A) + det(B)$.

(d) If $A^k = 0$ for some positive integer k, then A is not invertible.

(e) When a multiple of one row is subtracted from another, the eigenvalues do not change.

(f) Suppose that $\lambda \neq 0$ is an eigenvalue of A, and x is its eigenvector. Then x is also an eigenvector of A^{-1} corresponding to the eigenvalue $1/\lambda$.

2. [`5+3+4=12 pts`]

(a) Find a basis for the space of vectors that satisfy the following :

$$\begin{aligned} 2x_1 + 6x_2 - 2x_3 + 6x_4 &= 0 \\ x_1 + 3x_2 + 5x_4 &= 0 \\ 3x_1 + 9x_2 + 15x_4 &= 0 \end{aligned}$$

(b) What is the dimension of the nullspace in the above example?

(c) Find complete solutions for the following :

$$\begin{aligned} 2x_1 + 6x_2 - 2x_3 + 6x_4 &= 4 \\ x_1 + 3x_2 + 5x_4 &= 2 \\ 3x_1 + 9x_2 + 15x_4 &= 6 \end{aligned}$$

3. [`5×6=30 pts`] Consider the following symmetric matrix :

$$M = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) Find the triangular L and diagonal D such that $M = LDL^T$.

(b) Show that M is positive definite.

(c) What is the smallest number that could replace the element $M_{33} = 1$ and makes the eigenvalues of M be all nonnegative ?

(d) Show that the minimum value of $\frac{x^T M x}{x^T x}$ for all 3×1 nonzero vectors x is the smallest eigenvalue of M.

(e) Removing the last row and column of a positive definite matrix always leaves a positive definite matrix. Explain why.

4. [`20 pts`]

(a) Find the Fourier sine series of $f(x) = x$, $0 < x < L$.

(b) Solve for $u(x,t)$.

$$\begin{aligned} c^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, &\qquad& 0 < x < L,\ t > 0 \\ u(0,t) &= 0 && u(L,t) = 0 \\ u(x,0) &= x && \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \end{aligned}$$

5. [`20 pts`] Use the fact that

$$\mathcal{F}\{e^{-\frac{x^2}{4p^2}}\} = \sqrt{2}pe^{-p^2\omega^2}$$

to solve the following:

(a) Show that a solution to the following

$$\begin{aligned} k\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, && -\infty < x < \infty,\ t > 0 \\ u(x,0) &= e^{-x^2} && -\infty < x < \infty. \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{1+4kt}} e^{\frac{-x^2}{1+4kt}} .$$

[Hint] Fourier transform.

(b) Show that a solution to the following

$$\begin{aligned} k\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, && -\infty < x < \infty,\ t > 0 \\ u(x,0) &= f(x) && -\infty < x < \infty \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(\tau) e^{-(x-\tau)^2/4kt} d\tau .$$

[Hint] Convolution theorem.

- Fourier series for a periodic ftn with a period 2π :

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \; .$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx \end{aligned}$$

- Fourier integral

$$f(x) = \frac{1}{\pi} \int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \cos \omega x) \; d\omega$$

$$\begin{aligned} A(\omega) &= \int_{-\infty}^{\infty} f(x) \cos \omega x \; dx \\ B(\omega) &= \int_{-\infty}^{\infty} f(x) \sin \omega x \; dx \end{aligned}$$

- Fourier transform and its inverse :

$$\begin{aligned} \mathcal{F}(f) = \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \; dx \\ \mathcal{F}^{-1}(\hat{f}) = f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \; d\omega \\ \mathcal{F}(f * g) &= \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g) \end{aligned} \tag{1}$$