

2008 midterm

1. (2)

false.

$\text{rank}(A) = \text{rank}(A^T)$ for any mat.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{not square.}$$

(b)

$$A = P^{-1}BP.$$

$$A^{-1} = P^{-1}B^{-1}P.$$

\therefore true.

(c)

False.

$$A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad B = -A.$$

(d)

$$A^k = 0 \Rightarrow |A^k| = |A|^k = 0. \quad \therefore \text{true}$$

(e) False

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ vs } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$(\lambda \neq 0) \rightarrow A \text{ invertible.}$

$$(f) Ax = \lambda x \xrightarrow{(\lambda \neq 0)} A^{-1}Ax = \lambda x.$$

$$A^T y = \mu y \xrightarrow{\mu = \frac{c}{\lambda}} y = x. \quad \therefore \text{true}$$

$$2. \quad \left[\begin{array}{cccc|c} 2 & 6 & -2 & 6 & 4 \\ 1 & 3 & 0 & 5 & 2 \\ 3 & 9 & 0 & 15 & 6 \end{array} \right]$$

now elim.

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

(a) homo. sol : $\begin{cases} x_1 = -3x_2 - 5x_4 \\ x_3 = -2x_4 \end{cases}$

x_2, x_4 : free variables

(b) $x_h = c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

\nwarrow basis.

(b)
 $\dim(N(A)) = 2$.

(c) particular sol: $x_1 = 2, x_2, x_3, x_4 = 0$

$$x = x_h + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3.

$$(a) L_{32} L_{21} M = \begin{bmatrix} 3 & -1 & 0 \\ 0 & \frac{5}{3} & -1 \\ 0 & 0 & \frac{2}{5} \end{bmatrix},$$

where

$$L_{21} = \begin{bmatrix} 1 & & \\ \frac{1}{3} & 1 & \\ & & 1 \end{bmatrix}, \quad L_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & \frac{1}{3} & 1 \end{bmatrix}.$$

Thus

$$L = L_{32} L_{21} = \begin{bmatrix} 1 & & \\ \frac{1}{3} & 1 & \\ \frac{1}{9} & \frac{3}{5} & 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} 3 & & \\ & \frac{5}{3} & \\ & & \frac{2}{5} \end{bmatrix}.$$

- (b)
 - diagonal elements of $D =$ pivots of M are all positive. $\therefore M$ is pos def.
 - Or, determinant of upper-left $1 \times 1 \Rightarrow 3 > 0$
 $2 \times 2 \Rightarrow 5 > 0$
 $3 \times 3 \Rightarrow 2 > 0$
 $\therefore M$ is pos def.
 - Or, eigenvalues of $M : 0.267p, 2, 3.732i > 0$
 $\therefore M$ is pos def.
 - Or, $x^T M x = y^T D y$ where $y = L^T x$.
 $= d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2$ ($y \neq 0$ for $x \neq 0$)
 > 0 for $y \neq 0$.
 $\therefore M$ is pos def.

(c)

$$M = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & a \end{bmatrix} \xrightarrow{\text{row elimination}} \begin{bmatrix} 3 & -1 & 0 \\ \frac{5}{3} & -1 & 0 \\ 0 & -1 & a - \frac{3}{5} \end{bmatrix}$$

$a - \frac{3}{5} \geq 0$ for M to be pos semi def.

(d) Let X to be any eigenvector matrix of M .

And let $y = Xx$ for any nonzero vector x .
orthogonal

then

$$x^T x = y^T y, \quad x^T M x = y^T \Lambda y.$$

$$\therefore \frac{x^T M x}{x^T x} = \frac{\sum_{i=1}^3 \lambda_i y_i^2}{\sum_{i=1}^3 y_i^2}$$

$$\lambda_{\min} \leq \lambda_{\max}.$$

(e) $\therefore A^T C A$
 $\underbrace{n \times m \times m \times n}_{n \times n}$

- A must have n indept columns (full rank n)
- C must be pos def (so that $A^T C A$ is pos def for all those A).

(f) $K_1 \rightarrow K_2$ (without the last row & col.)

- $x^T K_2 x = [x^T \ 0] K_1 \begin{bmatrix} x \\ 0 \end{bmatrix} > 0$ for $x \neq 0$

- Or, one less pivot and determinant to test the pos. definiteness.

(In fact, $\lambda_{\min}(K_2) \geq \lambda_{\min}(K_1)$ where $\lambda_{\min}(K) = \min_{x \neq 0} \frac{x^T K x}{x^T x}$)