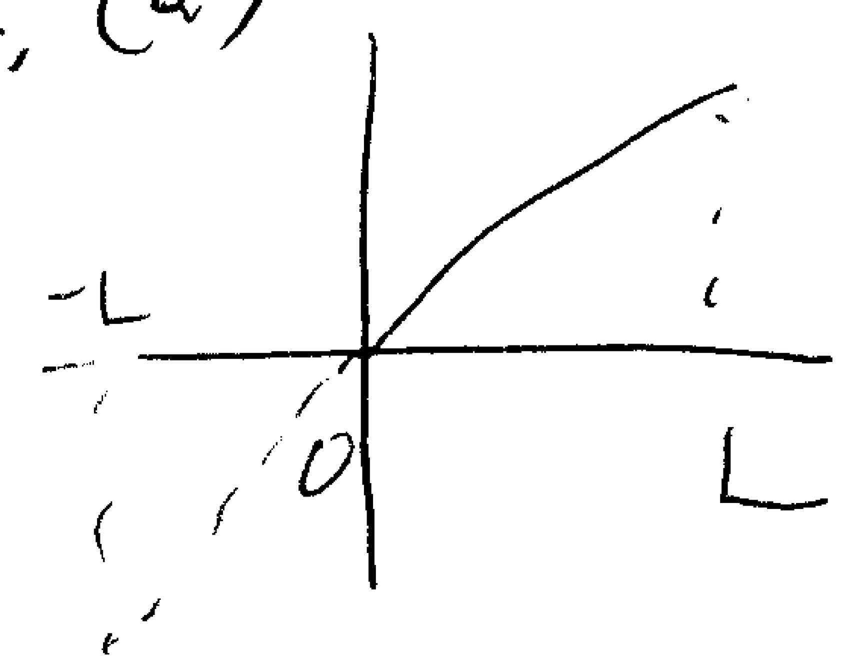
4. (a)



$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

$$b_n = \frac{2}{L}\int_0^L x \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi} x \cos \frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2 \sin \frac{n\pi x}{L} \Big|_0^L \right]$$

$$= \frac{2L}{n\pi}(-1)^{n+1}.$$

(b)

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(x,t) = X(x)T(t).$$

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -\lambda^2. \text{ (for nontriviality)}$$

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x \longrightarrow X(0) = X(L) = 0$$

$$\therefore c_1 = 0$$

$$\sin \lambda L = 0$$

$$\lambda_n = \frac{n\pi}{L},\ n = 1, 2, \dots$$

$$T = c_3 \cos \lambda ct + c_4 \sin \lambda ct$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n ct + B_n \sin \lambda_n ct) \sin \lambda_n x.$$

$$u(x,0) = x = \sum_{n=1}^{\infty} A_n \sin \lambda_n x.$$

↳ $A_n = \frac{2L}{n\pi}(-1)^{n+1}$ from (a).

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = \sum_{n=1}^{\infty} B_n \lambda_n c \sin \lambda_n x.$$

$$\hookrightarrow B_n = 0.$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \cos \lambda_n ct \sin \lambda_n x.$$

$$\left(\lambda_n = \frac{n\pi}{L}\right).$$

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5. $\mathcal{F}\left(e^{-\frac{x^2}{4p^2}}\right)$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4p^2}} e^{-i\omega x}\, dx = \sqrt{2}\, p\, e^{-p^2\omega^2}.$$

(a) $U(\omega,t) \triangleq \mathcal{F}(u(x,t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$

then $$-k\omega^2 U = \frac{dU}{dt}$$

$$\Rightarrow \quad U(\omega,t) = C e^{-k\omega^2 t}$$

$$U(\omega,0) = C = \mathcal{F}(e^{-x^2}) \underset{p=\frac{1}{2}}{=} \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}.$$

$$u(x,t) = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}} e^{-k\omega^2 t}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}(1+4kt)} e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}(1+4kt)} e^{-i\omega x}\, d\omega$$

$$\uparrow \quad p = \frac{1}{\sqrt{1+4kt}}.$$

$$= p e^{-p^2x^2} = \frac{1}{\sqrt{1+4kt}} e^{-\frac{x^2}{1+4kt}}.$$

5(b)

$$\mathcal{F}(u(x,t)) = U(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx.$$

$$-k\omega^2 U = \frac{dU}{dt}.$$

$$\Rightarrow U(\omega,t) = C e^{-k\omega^2 t}.$$

$$U(\omega,0) = C = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= F(\omega).$$

$$u(x,t) = \mathcal{F}^{-1}\left(e^{-k\omega^2 t} \cdot F(\omega)\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\, g(x-\tau)\, d\tau.$$

where $g(x) = \mathcal{F}^{-1}\{e^{-k\omega^2 t}\}$ $\quad p = \sqrt{kt}$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{kt}}\mathcal{F}^{-1}\left(\sqrt{2}\sqrt{kt}\, e^{-kt\omega^2}\right)$$

$$= \frac{1}{\sqrt{2}\sqrt{kt}} \cdot e^{-x^2/4kt}$$

$$\therefore u(x,t) = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(\tau)\, e^{-\frac{(x-\tau)^2}{4kt}}\, d\tau. \quad //$$