School of Mech & Aero Eng
Seoul National University

Eng Math 2
Fall '08

FINAL

- Do not open exam until told to do so.
- 'How you arrived at your answer' is much more important than the answer itself. Read carefully, and make sure you show your work *step by step*.
- Please use a separate sheet for each problem.
- Ask questions if you don't understand what the problem says.
- I thank you all, and wish you a wonderful winter break. But before that, **GOOD LUCK** tonight !

Student ID: ________________

Name: ________________

1	/ 10
2	/ 15
3	/ 15
4	/ 10
5	/ 10
6	/ 30
7	/ 10
Total	/ 100

1. [`10 pts`]

(a) Compute

$$\int_C \bar{z}\, dz\ , \quad C:\ y=(x/3)^2,\ -3 \leq x \leq 12$$

(b) True or false? Why?

(1) The function $f(z) = e^{1/(z-1)}$ has an essential singularity at $z = 1$.

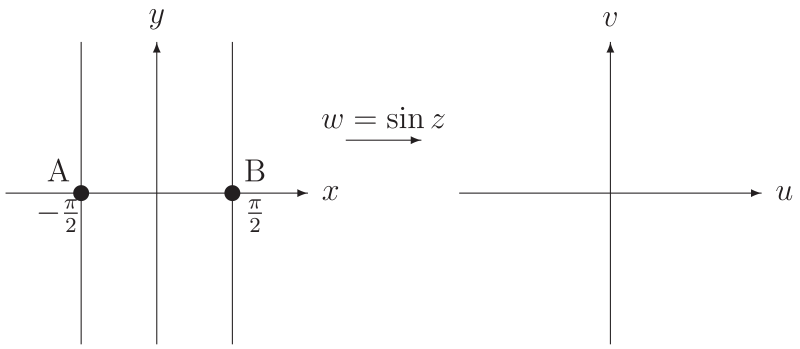
(2) If $w = f(z)$ is an analytic function that maps a domain D onto the upper half plane $v > 0$, then the function $u = Arg(f(z))$ is harmonic in D.

2. [`15 pts`] Solve for the temperature $u(x, y, t)$ in the plate such that, for $k > 0$,

$$
\begin{aligned}
k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \qquad & 0 < x < b,\ 0 < y < c,\ t > 0 \\
u(0, y, t) = 0,\ \ u(b, y, t) = 0 \qquad & 0 < y < c,\ t > 0 \\
u(x, 0, t) = 0,\ \ u(x, c, t) = 0 \qquad & 0 < x < b,\ t > 0 \\
u(x, y, 0) = f(x, y) \qquad & 0 < x < b,\ 0 < y < c
\end{aligned}
$$

3. [`15 pts`] Recall the definition of a conformal mapping and solve the following problem.

(a) Is $f(z) = \sin z = u + iv$ conformal in the interior of the fundamental region $-\frac{\pi}{2} < x < \frac{\pi}{2}$? Why do you think so?



(b) Under the mapping $w = \sin z$,

(i) find the image of the vertical lines $x = \pm\frac{\pi}{2}$,
and the image of the horizontal line between the points A and B.

(ii) show that the vertical lines $z = a + it$, $-\frac{\pi}{2} < a < \frac{\pi}{2}, a \neq 0$ are mapped to hyperbolas.

(iii) show that the horizontal lines $z = t + ib$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$ are mapped to the upper/lower part of ellipse depending on whether $b > 0$ or $b < 0$.

(c) Are the hyperbolas and ellipses found in (b) are orthogonal? Why do you think so?

4. [`10 pts`]

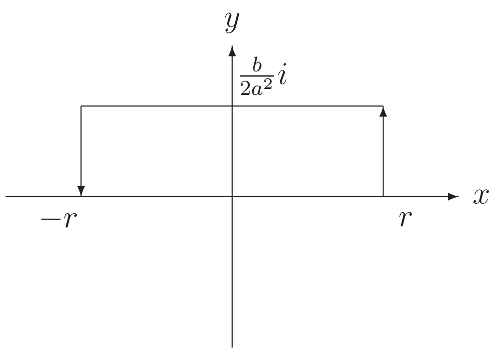
(a) Without integrating, find an upper bound for $\oint_C \frac{e^{2z}}{z+1} dz$ where C is a counterclockwise circle $|z| = 4$.

(b) Compute the above integral and check that your answer in (a) is true.

5. [`10 pts`]

Use the known result $\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \sqrt{\pi}/a$ and the following contour to show that

$$\int_0^\infty e^{-a^2x^2} \cos bx \; dx = e^{-b^2/4a^2} \sqrt{\pi}/2a \; .$$



6. [`30 pts`] Compute or show the following:

(a) $\oint_C \frac{1}{z(e^z-1)} dz, \quad C: |z| = 1$

(b) $\oint_C ze^{3/z} + \frac{\sin z}{z^2(z-\pi)^3} dz, \quad C: |z| = 5$

(c) $\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{\sqrt{(a^2-1)^3}}, \quad a > 1$

7. [`10 pts`] Expand $f(z) = \frac{z^2-2z+2}{z-2}$ in a Laurent series for the following domain:

(a) $1 < |z-1|$

(b) $0 < |z-2|$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$