4

(a) $$\left|\frac{e^{2z}}{z+1}\right| = \frac{|e^{2x}(\cos 2y + i\sin 2y)|}{|z+1|} = \frac{|e^{2x}|}{|z+1|}$$

$$\leq \frac{|e^{2x}|}{|z|-1} \leq \frac{e^{8}}{3} := M$$

$$L = 2\pi \cdot 4$$

$$ML = \frac{8\pi e^{8}}{3}.$$

(b) $$2\pi i \cdot \operatorname{Res}_{-1} \frac{e^{z^2}}{z+1} = 2\pi i \cdot e$$

5. b

$$\oint_C e^{-a^2(z^2 - \frac{ib}{a^2}z)}\,dz$$

$$= \oint_C e^{-a^2(z - \frac{b}{2a^2}i)^2}\,dz \cdot e^{-a^2\cdot\left(\frac{b}{2a^2}\right)^2}$$

$$e^{-a^2\cdot\left(\frac{b}{2a^2}\right)^2} = e^{-\frac{b^2}{4a^2}}.$$

$$= \Bigg[\int_{-r}^{r} e^{-a^2(x-\frac{b}{2a^2}i)^2}\,dx$$

$$+ \int_0^{\frac{b}{2a^2}i} e^{-a^2(r+y-\frac{b}{2a^2}i)^2}\,dy$$

$$+ \int_r^{-r} e^{-a^2x^2}\,dx$$

$$+ \int_{\frac{b}{2a^2}i}^{0} e^{-a^2(-r+y-\frac{b}{2a^2}i)^2}\,dy\Bigg] e^{-\frac{b^2}{4a^2}}$$

6. (a)

$$z(e^z-1) = z\left(z+\frac{z^2}{2}+\cdots\right).$$

$$\oint_C \frac{1}{z^2\left(1+\frac{z}{2}+\cdots\right)}\,dz = 2\pi i \times \left(-\frac{1}{2}\right)$$

$$= -\pi i.$$

$\cancel{f(}$

$$\mathrm{Res}(f,0) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{1+\frac{z}{2}+\cdots}\right)$$

$$= -\frac{1}{2}.$$

6(b)

$$z\left(1+\frac{3}{z}+\frac{1}{2!}\left(\frac{3}{z}\right)^2+\cdots\right)+\frac{\sin z}{z^2(z-\pi)^3}$$

$\downarrow$ $z=0$

$\downarrow$ $z=0.\ z=\pi.$

$$\hookrightarrow 2\pi i\cdot\frac{1}{2!}\cdot 9 = 9\pi i$$

$$\operatorname{Res}_{0} = \lim_{z\to 0}\frac{d}{dz}\frac{\sin z}{(z-\pi)^3} = \lim_{z\to 0}\left(-\frac{\sin z\cdot 3}{(z-\pi)^4}+\frac{\cos z}{(z-\pi)^3}\right)$$

$$= -\frac{1}{\pi^3}$$

$$\operatorname{Res}_{\pi} = \lim_{z\to\pi}\frac{1}{2}\cdot\frac{d^2}{dz^2}\frac{\sin z}{z^2}$$

$$= \lim_{z\to\pi}\cancel{\frac{1}{2}}\left(\frac{2}{\pi^3}\cdot\cancel{2}\right)$$

$$f' = \frac{-2\sin z}{z^3}+\frac{\cos z}{z^2}$$

$$f'' = \frac{6\sin z}{z^4}+\frac{-2\cos z}{z^3}+\frac{-2\cos z}{z^3}+\frac{-\sin z}{z^2}$$

$$\therefore 9\pi i + 2\pi i\left(-\frac{1}{\pi^3}+\frac{2}{\pi^3}\right)$$

$$= \frac{9\pi^3+2}{\pi^3}\, i.$$

//

6(c)

$$z = e^{i\theta}$$

$$\cos\theta = \frac{1}{2}(z + z^{-1})$$

$$d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{1}{(a+\cos\theta)^2} d\theta = \oint_C \frac{1}{iz} \frac{dz}{\left(a + \frac{1}{2}(z+z^{-1})\right)^2}$$

$$= \frac{4}{i} \oint_C \frac{z \quad dz}{(z^2 + 2az + 1)^2}$$

$$= \frac{4}{i} \oint_C \underbrace{\frac{z}{(z-z_0)^2 (z-z_1)^2}}_{f(z)} dz$$

$$\left(\begin{array}{l} z_0 = -a - \sqrt{a^2-1} \\ z_1 = -a + \sqrt{a^2-1} . \end{array} \right.$$

z_1 inside C.

$$= \frac{4}{i} \times 2\pi i \times \operatorname*{Res}_{z_1} f(z)$$

$$\operatorname*{Res}_{z_1} f(z) = \lim_{z \to z_1} \frac{d}{dz} \frac{z}{(z-z_0)^2}$$

$$= \lim_{z \to z_1} \left[\frac{1}{(z-z_0)^2} - \frac{2z}{(z-z_0)^3} \right]$$

$$= -\frac{z_1 + z_0}{(z_1 - z_0)^3} = \frac{a}{4\sqrt{(a^2-1)^3}}$$

$$= \frac{2\pi a}{\sqrt{(a^2-1)^3}}$$

↳ Ans. //

7. (a) $f(z) = z + \dfrac{2}{z-2}$

$= z + \dfrac{2}{(z-1)-1}$

$= z + \dfrac{2}{z-1} \cdot \left[\dfrac{1}{1 - \frac{1}{z-1}} \right]$

$= z + \dfrac{2}{z-1} \left(1 + \dfrac{1}{z-1} + \left(\dfrac{1}{z-1}\right)^2 + \cdots \right)$

(b) $(z-2) + 2 + \dfrac{2}{z-2}$.