SNU MAE Multivariable Control

Exam Fall 2008

Name:

1	/10
2	/15
3	/5
EXTRA	/α
Total	/30 +α

1. [10 pts]

(a) Prove that, for $x \in \mathbb{C}^n$ ($x \neq 0$) and $M \in \mathbb{C}^{m\times n}$,

$$\sqrt{\lambda_{min}(M^*M)} \leq \frac{||Mx||_2}{||x||_2} \leq \sqrt{\lambda_{min}(M^*M)}$$

where $||x||_2 = \sqrt{x^Tx}$.

(b) Describe why the following is true.
For a stable transfer matrix T such that $y(s) = T(s)u(s)$,

$$\sup_{u\neq 0} \frac{||y(t)||_2}{||u(t)||_2} = ||T||_\infty.$$

(a)

$$||Mx||_2^2 = x^* \underbrace{M^*M}_{n\times m\ m\times n} x$$

Hermitian. ≥ 0. $\Rightarrow$ ① real nonneg eval. ② transformation using orthogonal evec. matrix

$$= y^* \Lambda y \quad , \quad ||y||_2 = ||x||_2 \quad . \quad \Lambda = diag[\lambda_1, \cdots, \lambda_n]$$

($\lambda_1, \cdots, \lambda_n$: evals of M^*M)

$$\lambda_{min} ||y||_2^2 \leq \qquad \leq \lambda_{max} ||y||_2^2 .$$

(b) From (a),

for fixed ω,

$$\sup_{u(j\omega)} \frac{\|y(j\omega)\|_2^2}{\|u(j\omega)\|_2^2} = \bar{\sigma}\left(H(j\omega)\right)^2$$

$\downarrow$ H : stable.

and Parseval.

$$\sup_{u} \frac{\int_{-\infty}^{\infty} \|y(t)\|^2 dt}{\int_{-\infty}^{\infty} \|u(t)\|^2 dt} = \sup_{\omega} \bar{\sigma}\left(H(j\omega)\right)^2$$

$$= \|H\|_\infty$$

2. [15 pts] Consider the MIMO system with the 2×2 loop gain transfer matrix.

$$L(s) = \begin{bmatrix} 1/2 & 0 \\ ks/(s+2) & 0 \end{bmatrix}$$

The loop gain depends on a parameter $k \in \mathbb{R}$.

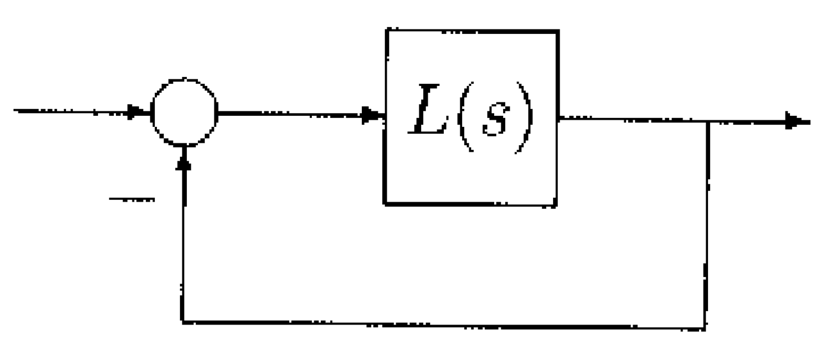
a) Find the ∞-norm of L (the norm still depends on k)

b) For which values of k does the small-gain theorem guarantee that the closed loop of Fig is internally stable?

c) Determine all k for which the closed loop is internally stable.

d) Compute $\mu(L(j\omega))$ with respect to structure

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}, \quad \Delta_{ij} \in \mathbb{C}.$$



(a)

$$L^*L = \begin{bmatrix} 1/2 & \frac{k(-j\omega)}{2-j\omega} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ \frac{kj\omega}{j\omega+2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} + \frac{k^2\omega^2}{2^2+\omega^2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$f'(\omega) = \frac{2\omega(4+\omega^2) - \omega^2(2\omega)}{(4+\omega^2)^2} > 0.$$

$$\therefore \max f(\omega) = \lim_{\omega\to\infty} f(\omega) = \frac{1}{4} + k^2.$$

$$\therefore -\frac{\sqrt{3}}{2} \le k \le \frac{\sqrt{3}}{2}.$$

(c) $\det(I+L) = 0$

$$= \begin{vmatrix} 1+\frac{3}{2} & 0 \\ \frac{ks}{s+2} & 1 \end{vmatrix}$$

$\frac{3}{2}$. always stable

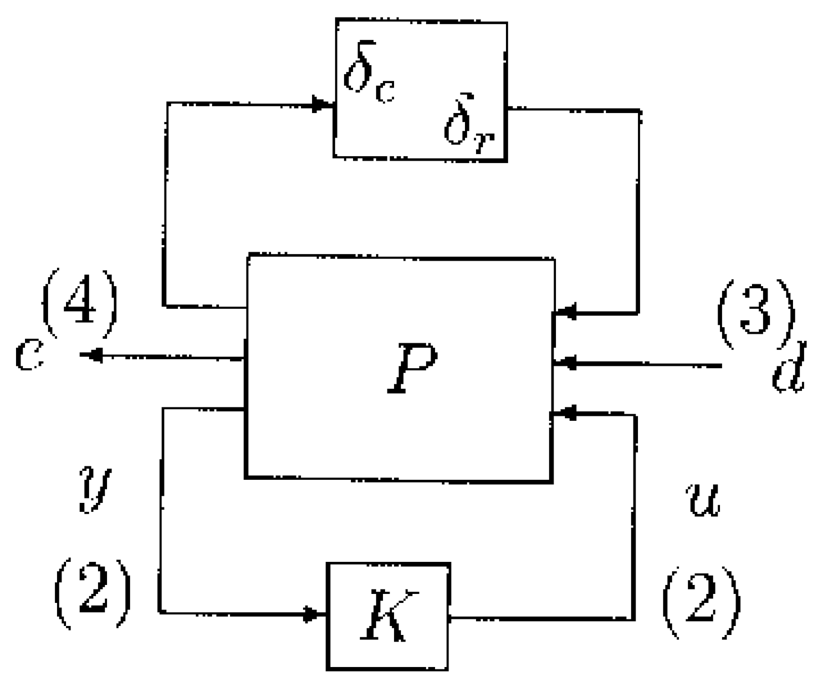
(d) $\det(I - M\Delta) = 0.$

$$M\Delta = \begin{pmatrix} \frac{1}{2}\Delta_{11} & \frac{1}{2}\Delta_{12} \\ \frac{ks}{s+2}\Delta_{11} & \frac{ks}{s+2}\Delta_{12} \end{pmatrix}$$

$$\mu_\Delta(L) = \bar{\sigma}(L) = \frac{1}{4} + \frac{k^2\omega^2}{2^2+\omega^2}$$

3. [5 pts] We can use D-K iteration for mu-synthesis. Then, describe why we still need to study H-infinity control ?

4. [EXTRA] Our plant has an uncertainty structure consisting of two scalar elements, one complex (δ_c) and the other real (δ_r). Our performance objective is a H_∞ norm from d to e. The signal d is 3-dimensional, and e is 4-dimensional. The controller takes 2-dimensional measurements and generates 2-dimensional control signals. And suppose we are given a stable controller K that achieves robust stability and robust performance.



Define a ball centered at K as

$$B(K) = \{\tilde{K} : \|\tilde{K} - K\|_\infty < 1, \tilde{K} : \text{ stable } 2 \times 2 \text{ transfer function}\}$$

and derive the necessary and sufficient condition such that every $\tilde{K} \in B(K)$ is a robustly performing controller. (It is a μ-statement, and you should explain the corresponding block structure and transfer function to be tested.)

4 perf obj: $\|T_{d\to e}\|_\infty$.

From the given condition on K,

K : stable

$$\max_\omega \mu_{\Delta_{aug}}(F_L(P,K)(j\omega)) < 1$$

where $\Delta_{aug} = \left\{ \begin{bmatrix} \delta_c & & \\ & \delta_r & \\ & & \Delta_F \end{bmatrix} : \delta_c \in \mathbb{C}, \delta_r \in \mathbb{R}, \Delta_F \in \mathbb{C}^{3\times 4} \right\}$.

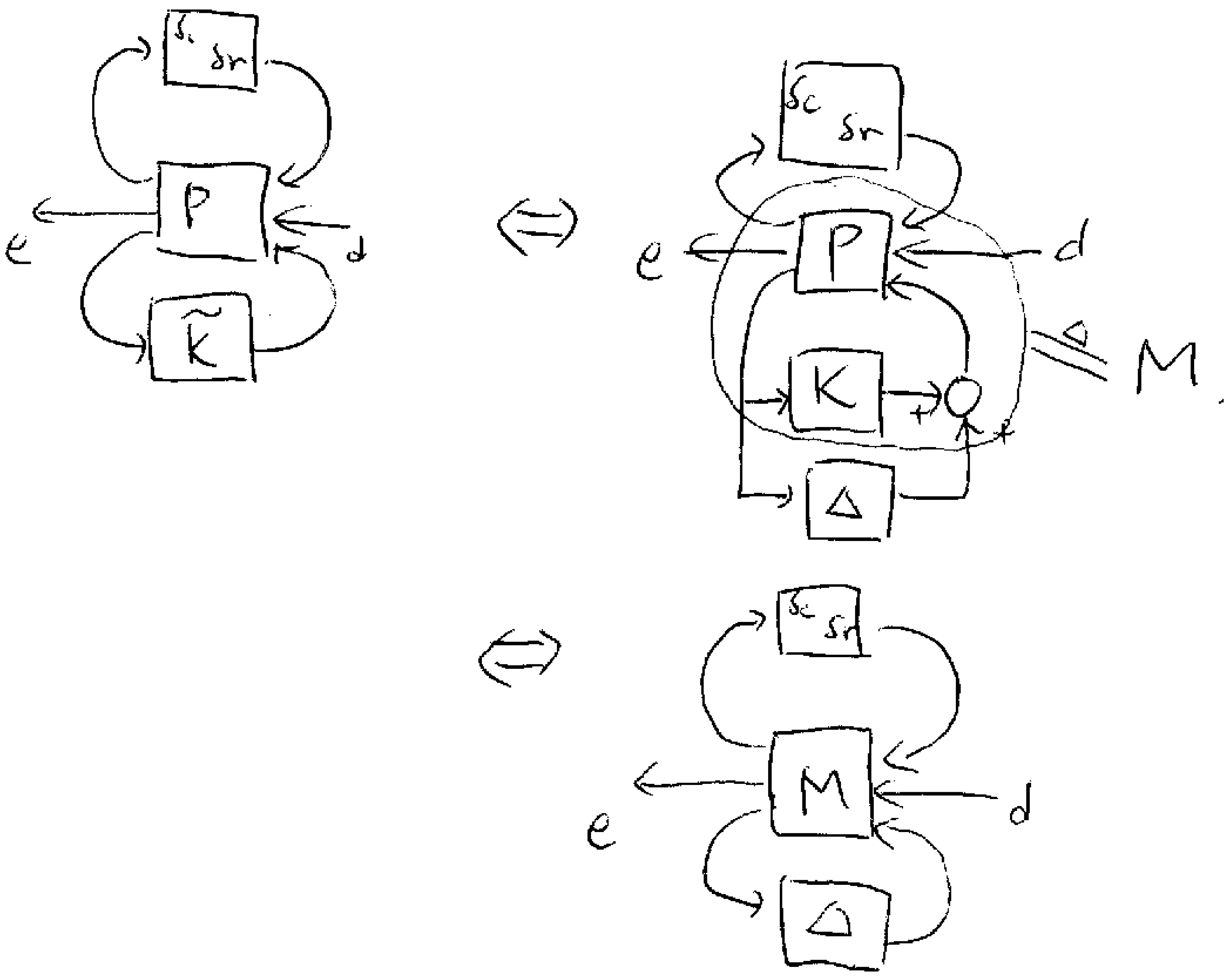
For each $\tilde{K} \in B(K)$, let

$$\Delta := \tilde{K} - K.$$

Then

$$\tilde{K} = K + \Delta, \quad \|\Delta\|_\infty \le 1, \quad \Delta \in \mathbb{C}^{2\times 2}.$$

Thus



i.e. rob. perf. test on $\tilde{K}$ is a μ test of M against

$\Delta_{new} = \left\{ \begin{bmatrix} \delta_c & & & \\ & \delta_r & & \\ & & \Delta_F & \\ & & & \Delta \end{bmatrix} : \delta_c \in \mathbb{C}, \delta_r \in \mathbb{R}, \Delta_F \in \mathbb{C}^{3\times 4}, \Delta \in \mathbb{C}^{2\times 2} \right\}$

∴ every $\tilde{K} \in B(K)$ is a robustly performing controller

iff

$$\max_{\omega} \mu_{\Delta_{new}}\left(M(j\omega)\right) < 1$$