97.6

BLUE BOOK

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※ 학칙을 위반하거나 학생의 본분에 어긋난 행위를 하였을 때에는 징계될 수 있습니다.

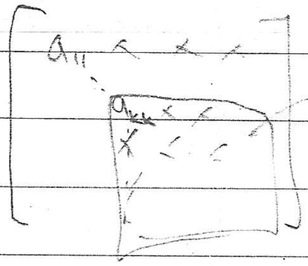
7 Pages

+3 Pages.

College of Engineering
Seoul National University

1.(a)

14.0



면적 = operation 수

$\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$

Gaussian Elimination 시에 Flops.

- multiplier 계산횟수, $\frac{n(n-1)}{2}$
- updating 횟수 (float 곱하기, 빼기) $\frac{n(n-1)(2n-1)}{6}$
- Right hand side → $\frac{n(n-1)}{2}$
- Back substitution → $\frac{n(n-1)}{2}$

rough 하게 보면 $\propto n^3$에 비례

8.5

$\left(\sqrt{10}^{\,3}\right)^3 = \sqrt{10}^{\,9}$

3D-particle diffusion equation $\vec{\nabla} \cdot \vec{J}(\vec{r}) + \sigma\phi(\vec{r}) = S(\vec{r})$

$-D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(x,y,z) + \sigma\phi(x,y,z) = S(x,y,z)$ → 2nd order accurate

mesh를 x, y, z 방향으로 m개씩 나누고 차분화 했을 때 나오는

linear system의 rank는 $m^3 = N$ 여기서 $m = 10 \times 2^k$ 라고 할때,

k를 한차수 증가시키면 rank는 $8m^3 = 8N = N'$

Flops 증가는 $N^3 \rightarrow (8N)^3 = 8\times8\times8\, N^3 = 512N^3$

즉, 512배 증가"

10 times accuracy enhancement → $\sqrt{10}$ times mesh ~~in~~ one dir

(b) 다양한 공학문제들에서 크기가 큰 sparse한 매트릭스를 갖는 linear system을 풀어야 하는 경우가 많이 있다. 이 linear system문제는 직접해법으로 풀수도 있고 간접해법으로 풀수도 있다. Gauss Elimination 이나 LU분해와 같은 직접해법으로 문제를 풀면 exact 한 답을 얻을수는 있지만, FLOPS가 $\propto n^3$에 비례하여 증가하는 문제점과 Elimination 시에 sparse한 메트릭스에 fill-in이 일어나는 문제점이 있다. 메모리 문제! 따라서 L-system의 사이즈가 커지는 경우에 직접해법의 방법은 효율적이지 못하다. 이에 비해 간접해법은 M^{-1}을 구하기 쉬운 $A = M + N$ 으로 분해하여 $Ax = b \rightarrow Mx^{(k)} = b - Nx^{(k-1)}$ 의 방식으로 해 x를 구하는 방식을 사용한다. 이 방식은 Iteration을 사용하여 해를 구하는 방법인데, 항상 수렴하는것이 아니라,

Iteration Matrix $\Rightarrow T = -M^{-1}N$ 의 eigenvalue 들의 크기가 1보다 작은 경우에 수렴한다. 공학문제에서는 문제의 해를 소수점 이하 부분까지 구할 필요는 없으므로 원하는 오차한계 내로 해가 들어올때까지만 Iteration을 시행하여 시간을 아낄수 있다. 또한 extrapolation 의 방법을 사용하는 Successive Over Relaxation 법을 사용하여 Spectral radius를 작게하여 수렴속도를 증가시킬수 있다. 반복해법에는 Jacobi법, Gauss-Seidel 법, SOR법 등이 있다.

(c) $$x^{(k)} = A\hat{x}^{(k-1)} \simeq \lambda_1^{(k)}\hat{x}^{(k-1)} \rightarrow \lambda_1^{(k)} = \frac{\langle x^{(k)}, x^{(k)}\rangle}{\langle x^{(k)}, \hat{x}^{(k-1)}\rangle}$$

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$$\hat{x}^{(k-1)} = \frac{x^{(k-1)}}{\lambda_1^{(k-1)}}$$

<Scaling 없이>

$$x^{(k)} = Ax^{(k-1)}$$

$$= A^k x^{(0)} \qquad x^{(0)} = c_1u_1 + c_2u_2 \cdots + c_nu_n \qquad u_i 는\ A의\ i번째\ \text{eigenvector}.$$

$$= A^k(c_1u_1 + c_2u_2 + \cdots + c_nu_n) \quad |\lambda_1| > |\lambda_2| > \cdots |\lambda_n|$$

$$= c_1\lambda_1^k u_1 + c_2\lambda_2^k u_2 + \cdots + c_n\lambda_n^k u_n$$

$$= \lambda_1^k\left[c_1u_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k u_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k u_n\right]$$

$$\simeq \lambda_1^k\left[c_1u_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k u_2\right]$$

$k\uparrow$

$$\simeq c_1\lambda_1^k u_1 \qquad \sigma = \frac{\lambda_2}{\lambda_1} \ \text{dominance ratio}$$

스케일링한 경우, $$x^{(k)} = \frac{\lambda_1^k}{\prod_{i=0}^{k-1}\lambda^i}\left[c_1u_1 + c_2\sigma^k u_2\right]$$

k가 커지면

$$\hat{x}^{(k)} \simeq c_1u_1 + c_2\sigma^k u_2$$

$$\hat{x}^{(k-1)} \simeq c_1u_1 + c_2\sigma^{(k-1)}u_2 \rightarrow \sigma^{k-1}$$

$$\hat{x}^{(k-2)} = c_1u_1 + c_2\sigma^{(k-2)}u_2 \rightarrow \sigma^{k-2}$$

$k-1$

$$e^{(k)} = \hat{x}^{(k)} - \hat{x}^{(k-1)} = c_2(\sigma^k - \sigma^{(k-1)})u_2$$

$$e^{(k-1)} = \hat{x}^{(k-1)} - \hat{x}^{(k-2)} = c_2(\sigma^{(k-1)} - \sigma^{(k-2)})u_2$$

$$\frac{\|e^{(k)}\|}{\|e^{(k-1)}\|} = \frac{c_2\|u_2\|\,\sigma^{(k-2)}(\sigma^2 - \sigma)}{c_2\|u_2\|\,\sigma^{(k-2)}(\sigma - 1)} = \sigma^{(k)}$$

9,5

따라서 dominance ratio의 추정은

$$\sigma^{(k)} = \frac{\|\hat{x}^{(k)} - \hat{x}^{(k-1)}\|}{\|\hat{x}^{(k-1)} - \hat{x}^{(k-2)}\|}$$

2. (a) $\rho(T) = \max_i |\lambda_i| \quad \forall i$ radius? definition?

15.0

$(M+N)x = b$

$Mx^{(k)} = b - Nx^{(k-1)}$, $Mx^* = b - Nx^*$ → 변변 빼면

$Me^k = -Ne^{(k-1)} \rightarrow e^{(k)} = \underbrace{-M^{-1}N}_{T}\, e^{(k-1)}$

$e^{(k)} = T e^{(k-1)}$

$= T^k e^{(0)}$ $\quad e^{(0)} = c_1u_1 + c_2u_2 \cdots + c_nu_n$

T의 eigenvector들의 combination

$= T^k(c_1u_1 + c_2u_2 + \cdots + c_nu_n)$ $\quad 1 > |\lambda_1| > |\lambda_2| \cdots > |\lambda_n|$

$= c_1\lambda_1^k u_1 + c_2\lambda_2^k u_2 + \cdots + c_n\lambda_n^k u_n$

$\simeq c_1\lambda_1^k u_1$ $\quad$ if $|\lambda_n| < |\lambda_{n-1}| < \cdots < |\lambda_2| < |\lambda_1| < 1$

$|\lambda_1| = \rho$ spectral radius

→ 작을수록 수렴이 빠르다.

~~9~~ 9.0

b) $\hat{x}^{(k-1)} = \dfrac{x^{(k-1)}}{\lambda^{(k-1)}}$

$x^{(k)} = A\hat{x}^{(k-1)} = \dfrac{A}{\lambda^{(k-1)}}x^{(k-1)} = \dfrac{A^2 x^{(k-2)}}{\lambda^{(k-1)}\lambda^{(k-2)}} \cdots$

$= \dfrac{A^k x^{(0)}}{\prod_{i=0}^{k-1}\lambda^i}$ $\quad x^{(0)} = c_1u_1 + c_2u_2 + \cdots + c_nu_n$

$= \dfrac{1}{\prod_{i=0}^{k-1}\lambda^i}\left(c_1\overset{\lambda_1^k}{A^k}u_1 + c_2\overset{\lambda_2^k}{A^k}u_2 + \cdots + c_n\overset{\lambda_n^k}{A^k}u_n\right)$

$= \dfrac{\lambda_1^k}{\prod_{i=0}^{k-1}\lambda^i}\left(c_1u_1 + c_2\left(\dfrac{\lambda_2}{\lambda_1}\right)^k u_2 + \cdots + c_n\left(\dfrac{\lambda_n}{\lambda_1}\right)^k u_n\right)$

k가 충분히 크면

$\simeq \dfrac{\lambda_1}{\prod_{i=0}^{k-1}\lambda^i}(c_1u_1)$ $\quad \lambda_1$ 에 대한 eigenvector로 수렴

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(c) Deflation method ; power method로 λ_1, u_1을 구했을 때

deflated matrix $B = A - \lambda_1\hat{u}_1\hat{u}_1^T$

$Bu_1 = Au_1 - \lambda_1\dfrac{u_1u_1^T}{u_1^Tu_1}u_1 = \lambda_1u_1 - \lambda_1u_1 = 0$

따라서 $x^{(k)} = Bx^{(k-1)} = B^k x^{(0)}$ $\rightarrow (c_1u_1 + c_2u_2 \cdots + c_nu_n)$

$= c_2\lambda_2^k u_2 + c_3\lambda_3^k u_3 + \cdots + c_n\lambda_n^k u_n$

$= \lambda_2^k\left(c_2u_2 + c_3\left(\dfrac{\lambda_3}{\lambda_2}\right)^k u_3 + \cdots + c_n\left(\dfrac{\lambda_n}{\lambda_2}\right)^k u_n\right) \simeq c_2\lambda_2^k u_2$

Decontamination method , symmetric matrix A의 eigenvector들은 orthogonal

$$x^{(k-1)} = c_1^{(k-1)} u_1 + c_2^{(k-1)} u_2 + \cdots + c_n^{(k-1)} u_n$$ 양변에 u_1 내적.

$$u_1^T x^{(k-1)} = c_1^{(k-1)} u_1^T u_1 + c_2^{(k-1)} \cancel{u_1^T u_2} + \cdots \; c_n^{(k-1)} \cancel{u_1^T u_n}$$

$$c_1^{(k-1)} = \frac{u_1^T x^{(k-1)}}{u_1^T u_1}$$

f1

따라서 decontaminated vector

$$\tilde{x}^{(k-1)} = x^{(k-1)} - \frac{u_1^T x^{(k-1)}}{u_1^T u_1} u_1$$

$$\hat{x}^{(k-1)} = \frac{\tilde{x}^{(k-1)}}{\lambda_2^{(k-1)}}$$

$$x^{(k)} = A \hat{x}^{(k-1)} \approx \lambda_2^{(k)} \hat{x}^{(k-1)} \rightarrow \lambda_2^{(k)} = \frac{\langle x^{(k)}, x^{(k)} \rangle}{\langle x^{(k)}, \hat{x}^{(k-1)} \rangle}$$

Deflation method 에서는 deflated matrix B를 만들때 A가 sparse 하다고 할지라도 B는 full matrix가 되는 치명적인 약점이 있다 따라서 일반적으로 연산횟수면에서 decontamination method가 더 효율적이지만 decontamination method는 각 k 단계 마다, $\hat{x}^{(k-1)}$을 구해줘야 하는 문제와 A가 Symmetric Matrix 일때에 한하여 효율적으로 이루어지는 한계가 있다.

~~10~~ 11

3.(a) Jacobi iteration $M = D \quad N = A - D$

~~12~~ 15.0

$$T = -M^{-1}N = -D^{-1}(A-D)$$

$$= [t_{ij}] = \left[-\frac{a_{ij}}{a_{ii}} \right] \leftarrow i \neq j$$

$i = j$ 일때 $t_{ii} = 0$

$$T u_j = \lambda u_j$$

$$\sum_{j \neq k} t_{kj} u_k = \lambda u_j \rightarrow$$ 곱의 절대값이 합의 절대값보다 작거나 같다.

$$|\lambda||u_j| \le \sum |t_{kj} u_k| < \sum |t_{kj}||u_j| = \sum \left|\frac{a_{kj}}{a_{kk}}\right| |u_j|$$

$$|\lambda| < \sum_{j \neq k} \left|\frac{a_{kj}}{a_{kk}}\right|$$ 에서 if $|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$ 이라면

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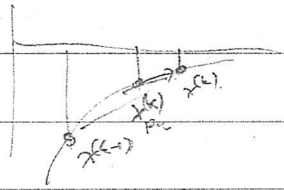
$$|\lambda| < \sum \left|\frac{a_{kj}}{a_{kk}}\right| < 1$$

□.

(b) 9.2번의 (b)번

(c)



$$x_i^{(k)} = \omega x_{i,GS}^{(k)} + (1-\omega)\, x_i^{(k-1)}$$

$$= \left[\omega D^{-1}\left(b - Lx^{(k)} - Ux^{(k-1)}\right) + (1-\omega)\,x^{(k-1)}\right]_i$$

$$Dx^{(k)} = \omega b - \omega L x^{(k)} - \omega U x^{(k-1)} + (1-\omega) D x^{(k-1)}$$

$$(D+\omega L)x^{(k)} = \omega b - \left(\omega U - (1-\omega)D\right)x^{(k-1)}$$

Identify $\underline{M = D + \omega L}$ $\quad \underline{N = \omega U - (1-\omega)D.}$ $\qquad$ 10

$$M + N = D + \omega L + \omega U - D + \omega D = \omega(L + D + U) = \omega A.$$

즉 $\omega A x = \omega b$

따라서 Iteration Matrix $T = -M^{-1}N = -(D+\omega L)^{-1}(\omega U - (1-\omega)D)$

4, (a) $x^{(k)} = \omega^{(k)} x^{(k)}_{pow} + (1-\omega^{(k)})x^{(k-1)}$ ← similar parameter with dependency on k

12.0

$$= \omega^{(k)}\left[\frac{A x^{(k-1)}}{\lambda^{(k-1)}}\right] + (1-\omega^{(k)})\,x^{(k-1)}$$

$$= \left[\omega^{(k)}\frac{A}{\lambda^{(k-1)}} + (1-\omega^{(k)})I\right] x^{(k-1)}$$

$$= \prod_{p=1}^{k}\left(\omega^{(p)}\frac{A}{\lambda^{(p-1)}} + (1-\omega^{(p)})I\right) x^{(0)} \qquad \text{if } x^{(0)} = c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

$$= \sum_{i=1}^{n} c_i \prod_{p=1}^{k}\left(\omega^{(p)}\frac{A}{\lambda^{(p-1)}} + (1-\omega^{(p)})I\right) u_i$$

$$= \sum_{i=1}^{n} c_i \underbrace{\prod_{p=1}^{k}\left(\omega^{(p)}\frac{\lambda_i}{\lambda^{(p-1)}} + (1-\omega^{(p)})\right)}_{\lambda_i\text{의 k차 다항식}} u_i$$

$\lambda : \lambda_n \sim \lambda_2$

$\gamma : -1 \sim 1$

$\gamma + 1 = 2 = \lambda - \lambda_n : \lambda_2 - \lambda_n$

$\gamma = \frac{2(\lambda_i - \lambda_n)}{\dots} - 1 \sim \frac{2(\lambda_i)}{\dots} - 1 \qquad \gamma_1 = \frac{2}{\dots} - 1 > 1$

$\lambda_i \to \gamma_i$ 변수변환. $\lambda_i = \frac{\gamma_i + 1}{2}\lambda_2$

$$x^{(k)} = \sum_{i=1}^{n} c_i \underbrace{\prod_{p=1}^{k}\left(\omega^{(p)} \frac{\lambda_2}{\lambda^{(p-1)}}\left(\frac{\gamma_i+1}{2}\right) + (1-\omega^{(p)})\right)}_{\eta_k(\gamma_i)} u_i$$

$\lambda^{(p-1)} \to \lambda_1 \quad \therefore \frac{\lambda_2}{\lambda_1} \to \sigma$ 10

b) Rationale?

$$= \sum_{i=1}^{n} c_i \eta_k(\gamma_i) u_i = c_1 \eta_k(\gamma_1) u_1 + \sum_{i=2}^{n} c_i \eta_k(\gamma_i) u_i$$

(b)

(c)

$$x^{(k)} = \eta_k(\gamma_1)\Big[c_1 u_1 + \underbrace{\sum_{i=2}^{n} c_i \frac{\eta_k(\gamma_i)}{\eta_k(\gamma_1)} u_i}_{\text{오차 term}}\Big]$$

오차 term의 $\eta_k(\gamma_i)$ 다항식을 chebyshev polynomial로 만들기

$$\eta_k(\gamma_i) = \prod_{p=1}^{k}\left(\omega^{(p)}\sigma\frac{\gamma_i+1}{2} + (1-\omega^{(p)})\right)$$

why? 4

(b)

$$\omega^{(p)}\sigma\frac{\xi_p+1}{2} + (1-\omega^{(p)}) = 0$$

ξ_p는 p번째 chebyshev polynomial 근

$$T_k(x) = \cos(k\theta) = \cos(k\cos^{-1}x)$$

$$k\theta = p\pi - \frac{\pi}{2} \qquad \theta = \frac{2p-1}{2k}\pi$$

$\frac{2p-1}{2}\pi$

$$x = \cos\left(\frac{2p-1}{2k}\pi\right) = \xi_p$$

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$$\omega^{(p)} = \frac{1}{1-\frac{\sigma}{2}(\xi_p+1)} = \frac{1}{1-\frac{\sigma}{2}\left(\cos\left(\frac{2p-1}{2k}\pi\right)+1\right)}$$

σ는 iteration 도중에 고유치 비율로 구함.

5.(a) $\lambda^{(0)} = 1$, $x^{(0)} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\hat{x}^{(0)} = \frac{x^{(0)}}{\lambda^{(0)}} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad x^{(k)} = A\hat{x}^{(k-1)}$

41.6

$$x^{(1)} = \begin{pmatrix}5 & -3 & -4\\ -3 & 5 & 0\\ -4 & 0 & 10\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\6\end{pmatrix}$$

$4+4+36$

$-2+2+6$

$$\lambda^{(1)} = \frac{\langle x^{(1)}, x^{(1)}\rangle}{\langle x^{(1)}, \hat{x}^{(0)}\rangle} = \frac{44}{6} = \frac{22}{3}$$

$$\hat{x}^{(1)} = \frac{x^{(1)}}{\lambda^{(1)}} = \frac{3}{22}\begin{pmatrix}-2\\2\\6\end{pmatrix} = \frac{3}{11}\begin{pmatrix}-1\\1\\3\end{pmatrix}$$

$$x^{(2)} = A\hat{x}^{(1)}$$

$$= \begin{bmatrix} 5 & -3 & -4 \\ -3 & 5 & 0 \\ -4 & 0 & 10 \end{bmatrix} \begin{bmatrix} -\frac{3}{11} \\ \frac{3}{11} \\ \frac{9}{11} \end{bmatrix}$$

$$-\frac{15}{11} - \frac{9}{11} - \frac{36}{11} \quad -\frac{60}{11}$$

$$\frac{9}{11} + \frac{15}{11} = \frac{24}{11}$$

$$\frac{12}{11} + \frac{90}{11} \quad \frac{102}{11}$$

$$= \begin{bmatrix} -\frac{60}{11} \\ \frac{24}{11} \\ \frac{102}{11} \end{bmatrix}$$

$$\left(\frac{60}{11}\right)^2 + \left(\frac{24}{11}\right)^2 + \left(\frac{102}{11}\right)^2$$

$$\frac{1}{11^2}\left(-3\times 60 + 3\times 24 + 9\times 102\right)$$

$$\lambda^{(2)} = \frac{\langle x^{(2)}, x^{(2)} \rangle}{\langle x^{(2)}, \hat{x}^{(1)} \rangle} = \frac{\frac{1}{11^2}(14580)}{\frac{1}{11^2}(810)} = \cancel{17.91}\ 12.46$$

$$\hat{x}^{(2)} = \frac{x^{(2)}}{\lambda^{(2)}} \doteq \begin{pmatrix} 0.3046 \\ 0.1218 \\ 0.5177 \end{pmatrix}$$

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(b) $A^{-1}x = \tilde{\lambda}x$

$$\det(A^{-1} - \tilde{\lambda}I) = \det(A^{-1}(I - \tilde{\lambda}A)) = \det\left(A^{-1}\left(-\frac{1}{\tilde{\lambda}}\right)\left(A - \frac{1}{\tilde{\lambda}}I\right)\right) = \ldots$$

$$\therefore \det\left(A - \frac{1}{\tilde{\lambda}}I\right) = 0$$

$$\therefore \frac{1}{\tilde{\lambda}} = \lambda$$

$x^{(k)} = A^{-1}\hat{x}^{(k-1)}$ → A^{-1} 구하기 힘들므로...

$A x^{(k)} = \hat{x}^{(k-1)}$ ← $Ax=b$와 같은형태로서 $A=LU$, $LUx=b$ 푼다.
안되면 반복법으로...

$$\tilde{\lambda}^{(k)} = \frac{\langle x^{(k)}, x^{(k)} \rangle}{\langle x^{(k)}, \hat{x}^{(k-1)} \rangle}$$ ← 여기서 구한 $\tilde{\lambda}$가 $\frac{1}{\lambda}$의 최대이므로 λ가 최소 eigenvalue

$$\lambda^{(k)} = \frac{\langle x^{(k)}, \hat{x}^{(k-1)} \rangle}{\langle x^{(k)}, x^{(k)} \rangle}$$ ← 이게 구하고자 하는 A의 최소 eigenvalue 근사

$$\left(\hat{x}^{(k-1)} = \frac{x^{(k-1)}}{\tilde{\lambda}^{(k-1)}} = \lambda^{(k-1)} x^{(k-1)}\right)$$

$$\lambda^{(k)} = \frac{\lambda^{(k-1)} \langle x^{(k)}, x^{(k-1)} \rangle}{\langle x^{(k)}, x^{(k)} \rangle}$$

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5. (c)

Householder matrix H

$$H = I - 2\hat{v}\hat{v}^T$$

여기서 $v = x + \alpha e$, $\alpha = \|x\|$

$Hx = -\alpha e$ .. 왜냐하면

$$v^T v = (x^T + \alpha e^T)(x + \alpha e)$$
$$= x^T x + \alpha x^T e + \alpha e^T x + \alpha^2 e^T e$$
$$= 2\alpha^2 + 2\alpha e^T x$$

$$v v^T x = (x + \alpha e)(x^T + \alpha e^T)x$$
$$= (x x^T + \alpha x e^T + \alpha e x^T + \alpha^2 e e^T)x$$
$$= x\underbrace{(x^T x)}_{\alpha^2} + \alpha x(e^T x) + \alpha e\underbrace{(x^T x)}_{\alpha^2} + \alpha^2 e(e^T x)$$
$$= \alpha^2 x + \alpha(e^T x)x + \alpha^3 e + \alpha^2(e^T x)e$$
$$= \alpha(\alpha + e^T x)x + \alpha^2(\alpha + e^T x)e$$
$$= \alpha(\alpha + e^T x)(x + \alpha e)$$

$$\therefore Hx = (I - 2\hat{v}\hat{v}^T)x$$
$$= x - 2\frac{v v^T x}{v^T v}$$
$$= x - \cancel{2}\frac{\cancel{\alpha(\alpha + e^T x)}}{\cancel{2}\cancel{\alpha(\alpha + e^T x)}}(x + \alpha e)$$
$$= -\alpha e$$

$$U_1 \equiv \begin{bmatrix} 1 & [\ 0^T\ ] \\ [0] & [\ H\ ] \end{bmatrix} \quad A_1 = \begin{bmatrix} a_{11} & [\ y^T\ ] \\ [x] & [\ B\ ] \end{bmatrix}$$

$H = H^{-1} = H^T$ 이고 $U^{-1} = U = U^T$

$A_2 = U_1^{-1} A_1 U_1$ 유사 변환, (eigenvalue invariant)

$$= \begin{bmatrix} [a_{11} + 0^T x] & [\ y^T + 0^T B\ ] \\ [0 a_{11} + Hx] & [\ 0 y^T + HB\ ] \end{bmatrix} \begin{bmatrix} 1 & [\ 0^T\ ] \\ [0] & [\ H\ ] \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & [\ y^T H\ ] \\ \begin{bmatrix} -\alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & [\ HBH\ ] \end{bmatrix}$$

$$A_3 = U_2^{-1} A_2 U_2 = U_2^{-1} U_1^{-1} A_1 U_1 U_2$$
$$\vdots$$
$$A_{n-1} = U_{n-2}^{-1} U_{n-1}^{-1} \cdots U_1 A \cdot U_1 U_2 \cdots U_{n-2}$$
$$= U^{-1} A U$$

반복적으로 적용하면

$$= \begin{bmatrix} a_{11} & \times & \times & \times & \times \\ -\alpha & \ddots & & \times & \times \\ & 0 & \ddots & & \times \\ & & & \ddots & \times \end{bmatrix}$$ Hessenberg Form

$$A = \begin{bmatrix} 5 & [-3\ \ -4] \\ \begin{bmatrix} -3 \\ -4 \end{bmatrix} & \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \end{bmatrix}$$

$x = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$, $\alpha = 5$. $v = x + \alpha e = \begin{pmatrix} -3 \\ -4 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

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$\hat{v} = \frac{1}{\sqrt{20}}\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ $4 + 16 = 20$

$$\hat{v}\hat{v}^T = \frac{1}{\sqrt{20}}\begin{pmatrix} 2 \\ -4 \end{pmatrix}\frac{1}{\sqrt{20}}(2\ \ -4)$$
$$= \frac{1}{20}2\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot 2(1\ \ -2)$$
$$= \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$2\hat{v}\hat{v}^T = \frac{2}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

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5. (c) 계속.

$H = I - 2\hat{v}\hat{v}^T$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1-\frac{2}{5} & \frac{4}{5} \\ \frac{4}{5} & 1-\frac{8}{5} \end{pmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$, $A = \begin{bmatrix} 5 & -3 & -4 \\ -3 & 5 & 0 \\ -4 & 0 & 10 \end{bmatrix}$

$U_1AU_1 = \begin{bmatrix} 5 & -3 & -4 \\ -5 & 3 & 8 \\ 0 & 4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$= \begin{bmatrix} 5 & -5 & 0 \\ -5 & \frac{41}{5} & -\frac{12}{5} \\ 0 & -\frac{12}{5} & \frac{34}{5} \end{bmatrix}$ ✓ Hessenberg form //

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5. (d)

$A = QR$

$Q^{-1}A = R$

$P_1 A_1^{\text{첫번째col}} = r_{11}$

$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} \\ 0 \\ 0 \end{bmatrix}$

$a_{11}\sin\theta + a_{21}\cos\theta = 0$

$a_{11}^2(1-\mu^2) = a_{21}^2\mu^2$

$\mu = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}} = \cos\theta$ ✓

$\sin\theta = \frac{-a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}}$

$r_{11} = \frac{a_{11}^2}{\sqrt{a_{11}^2+a_{21}^2}} + \frac{a_{21}^2}{\sqrt{a_{11}^2+a_{21}^2}}$

$= \frac{25+25}{\sqrt{25+25}} = \frac{50}{5\sqrt{2}}$

$= \frac{10\sqrt{2}}{\sqrt{2}\sqrt{2}} = 5\sqrt{2}$ //

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5, (e)

① A를 Hessenberg form으로 바꾼다. $\rightarrow H = U^{-1} A U$

② $H_k = Q_k R_k$ 로부터.

$Q_k^{-1} H_k = R_k$

$P_{n-2} P_{n-3} \cdots P_1 H_k = R_k$ ✓

$\underline{Q_k = P_1^T P_2^T \cdots P_{n-2}^T}$

③ $H_{k+1} = R_k Q_k$ ✓

④ Lower diagonal entry가 0이 될때까지 반복.

⑤ diagonal entry가 eigenvalue들, $\underline{\widetilde{\lambda}_k}$

: shift를 사용하면

② $\widetilde{H}_k = H_k - \lambda_{(k)} I \quad \lambda_{(k)} = H_k(n,n)$.

③ $H_{k+1} = R_k Q_k + \lambda_{(k)} I$

※ $H_{k+1} = Q_k^{-1} H_k Q_k$ 유사변환이므로 eigenvalue 변하지 않음.

$= Q_k^{-1}(Q_k R_k) Q_k = \underline{R_k Q_k}$

앞에서 구한 eigen value들을 $\widetilde{\lambda}_k$라고 할때 각각에 대한 eigenvector들을 구하기 위해 shift.

$(A - \widetilde{\lambda}_k I)\, x^{(1)} = x^{(0)}$

$x^{(1)} = (A - \widetilde{\lambda}_k I)^{-1} x^{(0)}$

$x^{(0)} = c_1 u_1 + c_2 u_2 \cdots c_n u_n$

$= (A - \widetilde{\lambda}_k I)^{-1} (c_1 u_1 + c_2 u_2 + \cdots + c_n u_n)$

$= \dfrac{c_1 u_1}{\lambda_1 - \widetilde{\lambda}_k} + \dfrac{c_2 u_2}{\lambda_2 - \widetilde{\lambda}_k} + \cdots + \dfrac{c_k u_k}{\lambda_k - \widetilde{\lambda}_k} + \cdots + \dfrac{c_n u_n}{\lambda_n - \widetilde{\lambda}_k}$

$\simeq \dfrac{c_k}{\lambda_k - \widetilde{\lambda}_k} u_k$ ✓

임의의 초기 벡터로부터 k번째 eigenvector, $\widetilde{\lambda}_k$를 사용하여 단 한번의 iteration으로도 충분한 정확도로 k번째 eigenvector를 구해 볼수 있다.

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