Convex optimization 중간시험 (2008. 12. 5)

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1. Use the separating hyperplane theorem to prove the theorem of alternatives for linear strict generalized inequalities w.r.t. a proper cone K: exactly one of the following two systems is feasible.

$$Ax \prec_K b \ (I)$$

$$\lambda \neq 0,\ \lambda \succeq_{K^*} 0,\ A^T\lambda = 0,\ \lambda^T b \leq 0 \ (II)$$

2. Consider the following set.

$$K_{\text{pol}} = \left\{x \in \mathbb{R}^{2k+1} | x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k}\right\}.$$

Prof. 'H' argued as follows to prove that K_{pol} is solid. Here, we denote $f_t = (1, t, t^2, \ldots, t^{2k})^T$ so that $x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k} = f_t^T x$.

" Suppose $f_t^T \bar{x} > 0$ for all $t \in \mathbb{R}$. We will show there is $\epsilon > 0$ such that $f_t^T(\bar{x} + \epsilon u) > 0$ for all u with $\|u\| = 1$. To do so, it suffices to show that there are constants $\rho > 0$ and τ such that $\left(\frac{f_t}{\|f_t\|}\right)^T \bar{x} \geq \rho$ and $\left(\frac{f_t}{\|f_t\|}\right)^T \bar{x} \leq \tau$. But, since the set of $F = \left\{\frac{f_t}{\|f_t\|} : t \in \mathbb{R}\right\}$ is compact, the maximum of $\left(\frac{f_t}{\|f_t\|}\right)^T \bar{x}$ is attained at $\frac{f_{\bar{t}}}{\|f_{\bar{t}}\|}$ for some $\bar{t} \in \mathbb{R}$. Then, since $\bar{x} \in K_{\text{pol}}$, we have $\left(\frac{f_t}{\|f_t\|}\right)^T \bar{x} > 0$, and, therefore, bounded below by a positive number $\rho > 0$.

Also $\cdots$."

(a) What flaw(s) does his arguments have?

(b) Provide a correct proof that K_{pol} is solid.

(c) Can you characterize the interior of K_{pol} (other than the definition itself)?

3. A matrix $X \in S^n$ is called copositive if $z^T X z \geq 0$ for all $z \geq 0$. Verify that the set of copositive matrices is a proper cone. Find its dual cone.

4. Suppose f is differentiable. Then f is convex if and only if $\mathbf{dom} f$ is convex and for all $x, y \in \mathbf{dom} f$,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ as $\tilde{f}(x) = \{g(x) | g \text{ is affine},\ g(z) \leq f(z)\ \forall\ z\}$. Show $f(x) = \tilde{f}(x)$ for $x \in \text{int dom} f$.

6. Prove or disprove.

a. The sum of quasiconvex functions is also quasiconvex.

b. A log-concave function is quasiconcave. Necessarily concave?

c. The sum of log-concave functions is log-concave.

7. Prove if $X \succeq Y \succeq 0$ then $\det X \geq \det Y \geq 0$.