Final Exam - Convex Optimization (2008. 12. 18)

Name: SIN: Score: ( ) / 100

1. (a) Consider the problem of minimizing a convex differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ over a convex set X. Then $x \in X$ is optimal if and only if

$$\nabla f(x)^T (y - x) \geq 0, \ \forall \, y \in X.$$

(b) Consider an optimization problem $\min \{f(x) | Ax = b\}$ where f is a convex differentiable function. Then a feasible solution $\bar{x}$ of $Ax = b$ is optimal if and only if there is ν such that $\nabla f(\bar{x}) = A^T \nu$.

2. (a) Show that $x^T x \leq yz$, $y \geq 0$, and $z \geq 0$ if and only if

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, \ y \geq 0, \text{ and } z \geq 0.$$

(b) Rewrite the following problem as an SOCP.

$$\max \left\{ \left(\prod_{i=1}^{m} (a_i^T x - b_i) \right)^{1/m} | Ax \geq b \right\},$$

where a_i^T is the ith row of A.

3. Suppose a vector optimization problem has an optimal point x^*. Show that x^* is an optimal solution of the associated scalarized problem for any $\lambda \succ_{K^*} 0$. Also show the converse.

4. Derive a dual problem for

$$\min \sum_{i=1}^{N} \|A_i x + b_i\|_2 + \frac{1}{2} \|x - x_0\|_2^2,$$

where, $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$. First introduce new variables $y_i = A_i x + b_i$.

5. Derive the KKT condition for the problem,

$$\begin{array}{c} \min \ \mathrm{tr} X - \log\det X \\ \text{sub. to } Xs = y, \end{array}$$

with the domain S_{++}^n, and $y, s \in R^n$ with $s^T y = 1$. Verify the optimal solution is

$$X^* = I + yy^T - \frac{1}{s^T s} ss^T.$$

6. Let f_0, f_1, …, $f_m : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that the function

$$p^*(u, v) = \inf \{f_0(x) | x \in \mathcal{D}, \ f_i(x) \leq u_i, \ i = 1, \ldots, m, \ Ax - b = v\}$$

is convex.