18.03 Differential Equation - Fall 2004

Problem Set One

Due by noon on Friday, September 17 in boxes in 2-106.

Each question carries 10 points.

Collaboration is encouraged on problem sets. If you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. You must turn in your own writeups of all problems, and if you do collaborate, please write on your solution sheet the names of the students you worked with.

1a. Find the general solution of $y^3\frac{dy}{dx} = (y^4+3)\cos x$

1b. Find the general solution of $(\tan x)\frac{dy}{dx} = y$

2a. Solve the initial value problem $\frac{dy}{dx} = \frac{4x-y}{x-6y}, y(1) = 1$

2b. Solve the initial value problem $(x^2+1)\frac{dy}{dx} + 3x^3y = 6x\exp(-\frac{3}{2}x^2), y(0) = 5$

3.Find the general solution of $\frac{dy}{dx} + 5xy = \frac{y\ln y}{x}$ by substituting $v = \ln y$

4a. Show that the substitution $v = ax + by + c$ transforms the differential equation $\frac{dy}{dx} = F(ax+by+c)$ into a separable equation.

4b. Using (a) or otherwise, find the general solution of $\frac{dy}{dx} = x^2+y^2+2xy+4x+4y+4$

5a. Find the general solution of $y\frac{dx}{dy} + (2y-3)x = 4y^4$ *(Treat y as independent variable and find x(y))*

5b. Solve the initial value problem $x\frac{dy}{dx} + 3y = 2x^5, y(2) = 3$

6.Suppose that you discover in your attic an overdue 18.03 textbook on which your grandfather owed a fine of 5 cents 100 years ago. If an overdue fine grows exponentially at a 5% annual rate compounded continuously, how much would you have to pay if you return the book today? *(hint: think of this problem as continuously compounded interest)*

7.A tank contains 1000 liters (L) of a solution consisting of 50 kg salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture - kept uniform by stirring - is pumped out at the same rate. How long will it be until only 3 kg of salt remains in the tank?

8a. (Torricelli's Law. See end of Sec. 1.4 of EP.) This problem concerns a water tank in the shape of an inverted cone of height 1 m. The opening angle of the cone is 90 degrees. so the radius at the top is also 1 m. The cone is initially full of water. At t=0 a small hole is cut horizontally across the cone 1 cm. above the apex, so the hole has radius 1 cm. The water begins to drain. How long will it take for all the water to drain from the tank? Hint: sensible numerical approximations can simplify things. Use g=980 cm/(sec)^2. The answer is t=900 sec. to a close approximation.

8b. What is the volume of the conical tank? What is the radius of a cylindrical tank of height 1 m. with same volume as the cone? If initially full, how long does the cylindrical tank take to drain through a hole of radius 1 cm. in its base?

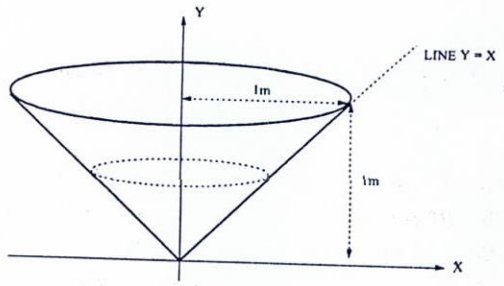

Figure 1: Water tank

9.Early one morning it began to snow at a constant rate. At 7 A.M. a snowplow set off to clear a road. By 8 A.M. it had travelled 2 miles, but it took two more hours (until 10 A.M.) for the snowplow to go an additional 2 miles.

a.Let $t = 0$ when it began to snow and let x denote the distance travelled by the snowplow at time t. Assuming that the snowplow clears snow from the road at a constant rate (in cubic feet per hour, say), show that $k\frac{dx}{dt} = \frac{1}{t}$ where k is a constant.

b.What time did it start snowing?

10. An advanced technology rocket expels fuel such that the total mass is $m(t)$, a smooth decreasing function of time. At time t, the velocity of the escaping fuel is $u(t)$ (relative to the rocket and directed opposite to its course), another given positive function. Both functions $m(t)$ and $u(t)$ are controlled from "Mission Control Center" on the ground. As a generalization of what was done in class, use the conservation of mass and momentum laws to derive the differential equation for the velocity:

$$m(t)\frac{dv(t)}{dt} = -u(t)\frac{dm(t)}{dt}$$

Show that, for constant $u(t) = u_0$, the solution to this ODE is

$$v(t) = v(0) + u_0 \ln\left(\frac{m(0)}{m(t)}\right)$$

18.03 Differential Equation - Fall 2004

Problem Set Two

Due by noon on Friday, September 24.

This homework is graded out of 100 marks. Each question carries 10 marks.

Collaboration is encouraged in this course. If you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. You must turn in your own writeups of all problems, and if you do collaborate, please write on your solution sheet the names of the students you worked with.

1a. Find the general solution of $\frac{dy}{dx} = y(xy^3 - 1)$

1b. Find the general solution of $x\frac{dy}{dx} - (1+x)y = xy^2$

2a. Find the general solution of $x\frac{dy}{dx} = y + x\exp\left(\frac{y}{x}\right)$

2b. Find the general solution of $\frac{dy}{dx} = \frac{x^2-y^2}{5xy}$

3. Solve the differential equation $\frac{dy}{dx} = \frac{x-y-1}{x+y+3}$ by finding h and k so that the substitution $x = u + h$, $y = v + k$ transform it into the homogeneous equation $\frac{dv}{du} = \frac{u-v}{u+v}$

4. Suppose that the population $P(t)$ of a country satisfies the differential equation $\frac{dP}{dt} = kP(200 - P)$ with constant k. Its population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

5a. Draw a few isoclines for the ODE $y' = 2y + x$.

5b. Use the isoclines drawn in (a) to sketch the direction field of the equation.

5c. Mark on the xy-plane the solution curve with initial condition $y(0) = 2$.

6a. Either use dfield6 or by hand, draw the direction field of the ODE $\frac{dy}{dx} = x^2 - y - 1$ for $-3 \leq x, y \leq 3$.

6b. Indicate on your plot the two solution curves that passes through $(0, -2)$ and $(0, 2)$.

7. Consider the ODE $y' = y(y - x)$ with initial condition $y(0) = y_0$. Solutions with $y_0 > y_{bif}$ seem to rise toward ∞ while those with $y_0 < y_{bif}$ rise a bit but fall toward 0. The value of y_{bif} is in range $0.790 < y_{bif} < 0.800$. Note that y_{bif} is the value of y_0 at the bifurcation point between rising and (eventually) falling solutions. In part b, c, you should express the solution in terms of a function defined by an integral.

7a. Use dfield6 to find y_{bif} to 4 significant figures.

7b. Solve the Bernoulli equation analytically and find the exact value of y_{bif}. Show all your reasoning. (Hint: $\int_0^\infty e^{-x^2/2}dx = \sqrt{(\pi/2)}$.

7c. Identify the separatrix, the curve which separates the xy-plane into regions, where solutions have the two distinct behaviors discussed above.

8. Suppose that a deer population $x(t)$ (in kilodeer) is modeled by the logistic equation $\frac{dx}{dt} = x(1 - x^2) = g(x)$.

8a. Sketch the function $g(x)$. (Allow both positive and negative x)

8b. Draw the phase line and sketch a few solutions of each type.

8c. Hunters are introduced into this system. They harvest the deer at a constant rate of h kilodeer per unit time. Describe what happens to the phase line as h increases from zero. (There is a critical value of h above which the population collapses to zero.)

9. Derive the solution $P(t) = \frac{MP_0}{P_0 + (M - P_0)\exp(-kMt)}$ of the logistic initial value problem $P' = kP(M - P)$, $P(0) = P_0$. Make it clear how your derivation depends on whether $0 < P_0 < M$ or $P_0 > M$.

10. Consider the first order autonomous ODE $x'(t) = (x - 10)(x + 10)^4(x + 20)$.

a. Find the critical points for the ODE.

b. Determine and sketch the phase diagram for this equation.

c. Sketch a few representative solution curves for this ODE (Exact solution is unnecessary).

DIRECTION FOR dfield6 ON ATHENA:

add matlab

matlab

(Matlab 7.0 Command window will open after a while)

>>DFIELD6

(Setup window appears)

Enter your ODE and set names and ranges of variables. (It seems that you must use t as independent variable.)

Click "Proceed" to get direction field in DFIELD6 Display Window.

Click on any point in Display to produce solution curve.

Click on options and test out "Keyboard input, Plot level curves, Plot several solutions" (caution on pss).

Engineering Mathematics I - Spring 2005

Problem Set 1

1 a Find the general solution of

$$y^3\frac{dy}{dx} = (y^4 + 3)\cos x$$

1 b Find the general solution of

$$\frac{dy}{dx}\tan x = y$$

2 a Solve the initial value problem

$$\frac{dy}{dx} = \frac{4x - y}{x - 6y}, \qquad y(1) = 1$$

2 b Solve the initial value problem

$$(x^2 + 1)\frac{dy}{dx} + 3x^3 y = 6x\exp(-\frac{3}{2}x^2), \qquad y(0) = 5$$

3 Find the general solution of

$$\frac{dy}{dx} + 5xy = \frac{y\ln y}{x}$$

by substituting $v = \ln y$

4 Find the general solution of

$$\frac{dy}{dx} = x^2 + y^2 + 2xy + 4x + 4y + 4$$

5 a Find the general solution of

$$y\frac{dx}{dy} + (2y - 3)x = 4y^4$$

(Treat y as independent variable and find $x(y)$)

5 b Solve the initial value problem

$$x\frac{dy}{dx} + 3y = 2x^5, \qquad y(2) = 3$$

6 a Find the general solution of

$$\frac{dy}{dx} = y(xy^3 - 1)$$

6 b Find the general solution of

$$x\frac{dy}{dx} - (1 + x)y = xy^2$$

7 a Find the general solution of

$$x\frac{dy}{dx} = y + x\exp(\frac{y}{x})$$

7 b Find the general solution of

$$\frac{dy}{dx} = \frac{x^2 - y^2}{5xy}$$

8 A tank contains 1000 liters (L) of a solution consisting of 50 kg salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture - kept uniform by stirring - is pumped out at the same rate. How long will it be until only 3 kg of salt remains in the tank?

9 Suppose that you discover in your backpack an overdue engineering mathematics textbook on which your friend owed a fine of 5 won 100 days ago. If an overdue fine grows exponentially at a 5% daily rate compounded continuously, how much would you have to pay if you return the book today?

10 Suppose that the population $P(t)$ of a country satisfies the differential equation $\frac{dP}{dt} = kP(200-P)$ with constant k. Its population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

Engineering Mathematics I - Spring 2005

Sample Midterm Test

April 21, 2005

1 Solve the following differential equations. An implicit solution is acceptable.

a

$$(x+y)y' = y$$

b

$$\frac{dy}{dx} = \frac{2\ln y + x}{\sin y - 2x/y}$$

2 Find the solution to the initial value problem

$$\frac{dy}{dt} + y = t \qquad y(0) = 0$$

3

a Solve the following:

$$\frac{dy}{dx} = \frac{1-3x^2-y^2}{2xy}$$

b Find the solution curve of this ODE that passes through the point (1,0). Sketch the solution curve on the x-y plane. What is the geometrical shape of this curve?

4 Solve the initial value problem

$$y'' + 4y' + 5y = 0 \qquad y(0) = 2, \quad y'(0) = 0$$

5 Write next to each of the following ODE's

(1) $y'' + y' - 6y = 0$

(2) $y'' + 5y' + 6y = 0$

(3) $y'' + 5y' + 6y = t^2$

(4) $y'' - 2y' + 5y = e^{-t}$

the letter which best describes the solutions:

A. All solutions tend to 0 as $t \to +\infty$.

B. At least one solution tends to 0 as $t \to +\infty$ but not all solutions have this property.

C. None of the above.

6 Knowing that $x/2 + x^2$ and $x/2 - 3x^3$ are two particular solutions of

$$x^2y'' + P(x)y' + 6y = x,$$

find $P(x)$. What is the homogeneous solution to this equation?

7 Find the general solution of $x^2y'' + 4xy' - 4y = x^2$.

8 Solve the initial problem $y'' + \pi^2 y = f(t)$ for the source

$$f(t) = \begin{cases} \pi^2 t & \text{for } 0 \le t \le 1 \\ \pi^2(2-t) & \text{for } 1 \le t \le 2 \\ 0 & \text{for } t > 2 \end{cases}$$

with initial conditions $y(0) = 0$, $y'(0) = 0$.

9 Find the general solution of the system

$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} e^{-t} \\ e^{t} \end{pmatrix}$$

10 Consider the nonlinear system:

$$\frac{dx}{dt} = x^2 - y$$

$$\frac{dy}{dt} = (x-1)^2 + y - 1$$

a Find all critical points of the system.

b At each critical point, linearize the system and indicate the type of the critical point.

c Sketch the phase portrait of the system in the xy-plane.

Engineering Mathematics I - Spring 2005

Midterm Examination

April 25, 2005

1

1a (10 points) Solve the initial value problem

$$\frac{dy}{dx} + \frac{2y}{x} = x^3, \qquad y(1) = 1$$

1b (10 points) Find the general solution of

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

1c (10 points) Find the general solution of

$$\frac{dy}{dx} + 2xy = 1 + x^2 + y^2$$

by using the substitution $y = x + \frac{1}{v}$. *Hint*: Remember that v is a function of x.

2

2a (10 points) Solve the initial value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 12y = 0, \qquad y(0) = y'(0) = 1$$

2b (10 points) Find the general solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}$$

2c (10 points) Consider the ordinary differential equation $y'' + 2y' + ky = 0$. For what values of k will the solutions be oscillatory, i.e. of the form $e^{at}\cos\omega t$, $e^{at}\sin\omega t$?

2d (10 points) A differential equation of the form $y'' + p(x)y' + q(x)y = 0$ is known to have $y_1(x) = x$ and $y_2(x) = e^x$ as solutions. Find $p(x)$ and $q(x)$.

3 For the nonlinear system

$$\frac{dx}{dt} = x - y$$

$$\frac{dy}{dt} = -y + 2x^2 - xy$$

3a (10 points) find the two critical points.

3b (10 points) At each critical point, linearize the system and indicate the *type* and *stability* of the critical point.

3c (10 points) Sketch the phase portrait around the two critical points in the xy-plane.

Engineering Mathematics I - Spring 2005

Examination 2

Total 150 points

June 14, 2005

1: 15 pts

Find a power series solution in powers of x of the following differential equation:

$$y'' - 4xy' + (4x^2 - 2)y = 0$$

2: 15 pts

Recalling that the standard form of Bessel's differential equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

find a general solution of the following equation in terms of Bessel functions. (Use the indicated substitutions.)

$$y'' + k^2x^4y = 0 \qquad (y = u\sqrt{x}, \quad \frac{1}{3}kx^3 = z)$$

3

(a: 10 pts) Find the Fourier sine series $\sum b_n \sin n\pi t$ for $1 - t$, on the interval (0,1].

(b: 10 pts) Find a Fourier since series solution $y(t)$ satisfying on (0,1) the ODE (k constant) $y'' + ky = 1 - t$, and the conditions $y(0) = 0$, $y(1) = 0$.

(c: 3 pts) Which term in the Fourier series solution in (b) would dominate if $k = 90$?

4: 15 pts

Solve the IVP $y'' - 2y' + y = 0$, $y(0) = 1$, $y'(0) = 0$ using the Laplace transform. No credit for other methods. You may use the table in the last page.

5

(a: 5 pts) Derive directly from the definitions the formula for the Laplace transform

$\mathcal{L}(f(t))$, where $f(t) = t$. (State explicitly any limit you use, but you need not prove it.)

(b: 5 pts) Let $F(s) = \mathcal{L}(f(t))$, and $a > 0$. Find a formula for $\mathcal{L}(f(at))$ in terms of $F(s)$.

6

(a: 10 pts) Solve by the Laplace transform: $y' = y - c\delta(t-1)$, $y(0) = 1$, where c is a constant and $\delta(t)$ is the Dirac delta function. Express your answer in the "cases" format,

$$y = \begin{cases} f(t), & 0 \leq t < a \\ g(t), & t \geq a \end{cases}$$

(b: 2 pts) For what value of the constant c will the solution be 0 for large t?

7

Consider the function $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin(t-\pi), & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

(a: 5 pts) Use step functions to give an alternate description of this function.

(b: 5 pts) Use the result of part (a) or another method to compute the Laplace transform of $f(t)$.

8

(a: 15 pts) In Los Angeles, a bicycle race is held every year to climb the steepest hill in the city. On the hill, the elevation is written as $f(x,y) = 4 - \frac{2}{3}\sqrt{x^2+y^2}$, $0 \leq z \leq 4$. Mathematically derive the direction of the shortest path to reach the apex of the hill using the concept of gradient. (5 points for a mere physical argument without using gradient)

(b: 5 pts) Find a unit normal vector for the following surface at the given point:

$$x^2 + y^2 + 2z^2 = 26, \qquad P : (2, 2, 3)$$

9

Evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dA$.

(a: 15 pts) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, $\quad S : \mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + 3v\mathbf{k}$, $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

(b: 15 pts) $\mathbf{F} = [x^3, y^3, z^3]$, $\quad S$: the sphere $x^2+y^2+z^2 = 9$. Use the divergence theorem.

Table of Laplace Transforms

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$
$e^{at} f(t)$	$F(s-a)$
f'	$sF(s) - f(0)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}