18.03 Differential Equation - Fall 2004

Problem Set One

Due by noon on Friday, September 17 in boxes in 2-106.

Each question carries 10 points.

Collaboration is encouraged on problem sets. If you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. You must turn in your own writeups of all problems, and if you do collaborate, please write on your solution sheet the names of the students you worked with.

1a. Find the general solution of $y^3\frac{dy}{dx} = (y^4+3)\cos x$

1b. Find the general solution of $(\tan x)\frac{dy}{dx} = y$

2a. Solve the initial value problem $\frac{dy}{dx} = \frac{4x-y}{x-6y}, y(1) = 1$

2b. Solve the initial value problem $(x^2+1)\frac{dy}{dx} + 3x^3y = 6x\exp(-\frac{3}{2}x^2), y(0) = 5$

3.Find the general solution of $\frac{dy}{dx} + 5xy = \frac{y\ln y}{x}$ by substituting $v = \ln y$

4a. Show that the substitution $v = ax + by + c$ transforms the differential equation $\frac{dy}{dx} = F(ax+by+c)$ into a separable equation.

4b. Using (a) or otherwise, find the general solution of $\frac{dy}{dx} = x^2 + y^2 + 2xy + 4x + 4y + 4$

5a. Find the general solution of $y\frac{dx}{dy} + (2y-3)x = 4y^4$ *(Treat y as independent variable and find $x(y)$)*

5b. Solve the initial value problem $x\frac{dy}{dx} + 3y = 2x^5, y(2) = 3$

6.Suppose that you discover in your attic an overdue 18.03 textbook on which your grandfather owed a fine of 5 cents 100 years ago. If an overdue fine grows exponentially at a 5% annual rate compounded continuously, how much would you have to pay if you return the book today? *(hint: think of this problem as continuously compounded interest)*

7.A tank contains 1000 liters (L) of a solution consisting of 50 kg salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture - kept uniform by stirring - is pumped out at the same rate. How long will it be until only 3 kg of salt remains in the tank?

8a. (Torricelli's Law. See end of Sec. 1.4 of EP.) This problem concerns a water tank in the shape of an inverted cone of height 1 m. The opening angle of the cone is 90 degrees. so the radius at the top is also 1 m. The cone is initially full of water. At t=0 a small hole is cut horizontally across the cone 1 cm. above the apex, so the hole has radius 1 cm. The water begins to drain. How long will it take for all the water to drain from the tank? Hint: sensible numerical approximations can simplify things. Use g=980 cm/(sec)^2. The answer is t=900 sec. to a close approximation.

8b. What is the volume of the conical tank? What is the radius of a cylindrical tank of height 1 m. with same volume as the cone? If initially full, how long does the cylindrical tank take to drain through a hole of radius 1 cm. in its base?

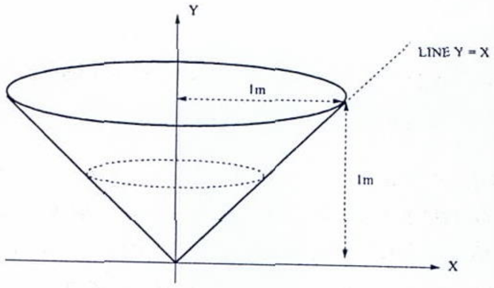

Figure 1: Water tank

9. Early one morning it began to snow at a constant rate. At 7 A.M. a snowplow set off to clear a road. By 8 A.M. it had travelled 2 miles, but it took two more hours (until 10 A.M.) for the snowplow to go an additional 2 miles.

a. Let $t = 0$ when it began to snow and let x denote the distance travelled by the snowplow at time t. Assuming that the snowplow clears snow from the road at a constant rate (in cubic feet per hour, say), show that $k\frac{dx}{dt} = \frac{1}{t}$ where k is a constant.

b. What time did it start snowing?

10. An advanced technology rocket expels fuel such that the total mass is $m(t)$, a smooth decreasing function of time. At time t, the velocity of the escaping fuel is $u(t)$ (relative to the rocket and directed opposite to its course), another given positive function. Both functions $m(t)$ and $u(t)$ are controlled from "Mission Control Center" on the ground. As a generalization of what was done in class, use the conservation of mass and momentum laws to derive the differential equation for the velocity:

$$m(t)\frac{dv(t)}{dt} = -u(t)\frac{dm(t)}{dt}$$

Show that, for constant $u(t) = u_0$, the solution to this ODE is

$$v(t) = v(0) + u_0 \ln(\frac{m(0)}{m(t)})$$

18.03 (fall 2004) Pset 1 solution

1)a) $y^3\frac{dy}{dx}=(y^4+3)\cos x$

$\int\frac{y^3}{y^4+3}dy=\int\cos x\,dx$

$\frac{1}{4}\ln(y^4+3)=\sin x+C$

$y^4=Ae^{4\sin x}-3$

1)b) $\tan x\frac{dy}{dx}=y$

$\int\frac{1}{y}dy=\int\cot x\,dx$

$\ln|y|=\ln|\sin x|+C$

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2)a) $u=y/x$

$\frac{dy}{dx}=u+x\frac{du}{dx}$

$x\frac{du}{dx}+u=\frac{4-u}{1-6u}$

$-\frac{1}{2}\int\frac{6u-1}{3u^2-u+2}du=\int\frac{1}{x}dx$

$-\frac{1}{2}\ln|3u^2-u+2|=\ln|x|+C$

$y(1)=1\Rightarrow C=-\frac{1}{2}\ln 4$

$-\frac{1}{2}\ln|3(\frac{y}{x})^2-(\frac{y}{x})+2|=\ln|x|-\frac{1}{2}\ln 4$

2)b) $\frac{dy}{dx}+\frac{3x(x^2+1)-3x}{x^2+1}y=6x\frac{\exp(-\frac{3}{2}x^2)}{(x^2+1)^{3/2}}$

$I=\exp(\int(3x-\frac{3x}{x^2+1})dx)=\frac{\exp(\frac{3}{2}x^2)}{(x^2+1)^{3/2}}$

2)b) $\frac{d}{dx}ye^{3/2x^2}(x^2+1)^{-3/2}=6x(x^2+1)^{-5/2}$

$ye^{3/2x^2}(x^2+1)^{-3/2}=-2(x^2+1)^{-3/2}+C$

$y(0)=5\Rightarrow C=7$

$ye^{3/2x^2}=\ \ -2\ \ +\ \ (x^2+1)^{3/2}$

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3) $\frac{dv}{dx}=\frac{1}{y}\frac{dy}{dx}$

$\frac{dv}{dx}+5x=\frac{v}{x}$

$\frac{dv}{dx}-\frac{1}{x}v=-5x$

$I=e^{\int-\frac{1}{x}dx}=\frac{1}{x}$

$\frac{d}{dx}\frac{v}{x}=-5$

$v/x=-5x+C$

$\ln y=-5x^2+Cx$

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4)a) $\frac{dv}{dx}=a+b\frac{dy}{dx}$

$\frac{dy}{dx}=F(ax+by+c)$

$\frac{1}{b}\frac{dv}{dx}-\frac{a}{b}=F(v)$

$\frac{dv}{dx}=bF(v)+a$

$\int\frac{dv}{F(v)b+a}=\int dx$

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4)b) $\frac{dy}{dx}=(x+y+2)^2$

Take $F(v)=F(x+y+2)=v^2$

where $v=x+y+2$

By (a), $\int\frac{dv}{v^2+1}=\int dx$

$\tan^{-1}v=x+C$

$\tan^{-1}(x+y+2)=x+C$

5a) $\frac{dx}{dy}+(2-\frac{3}{y})x=4y^3$

$I=\exp(\int(2-\frac{3}{y})dy)=e^{2y}/y^3$

$\frac{d}{dy}(x\cdot\frac{e^{2y}}{y^3})=4e^{2y}$

$\frac{x}{y^3}e^{2y}=2e^{2y}+C$

5)b) $x\frac{dy}{dx}+3y=2x^5$

$\frac{dy}{dx}+\frac{3}{x}y=2x^4$

$I=\exp(\int\frac{3}{x}dx)=x^3$

$\frac{d}{dx}yx^3=2x^7$

$yx^3=\frac{1}{4}x^8+C$

$y(2)=3\Rightarrow C=-40$

$yx^3=\frac{1}{4}x^8-40$

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6) $\frac{dX}{dt}=0.05X$

$X=Ae^{0.05t}$

$X(0)=A=0.05$

$X=0.05e^{0.05t}$

$X(100)=7.42$

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7) $\frac{dX}{dt}=-\frac{X}{1000}5$

$-200\ln X=t+C$

$X=Ae^{-\frac{t}{200}}$

$X(0)=A=50\Rightarrow X=50e^{-\frac{t}{200}}$

$X(t_1)=50e^{-\frac{t_1}{200}}=3\Rightarrow t_1=200\ln\frac{50}{3}$

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8)a) $A(y)=\pi y^2$

By (24) in Section 1.4 of EP

$\pi y^2\frac{dy}{dt}=-\pi(0.01)^2((2)(9.8)y)^{1/2}$

$\frac{2}{5}y^{5/2}=-0.01^2\sqrt{19.6}t+C$

$y(0)=1\Rightarrow C=2/5$

$2/5\,y^{5/2}=-0.01^2\sqrt{19.6}t+2/5$

$y=0\Rightarrow t\simeq 403.53$

8)b) $\frac{1}{3}\pi(1)^2(1)=\pi/3$

$\pi R^2h=\pi/3\Rightarrow R=1/\sqrt{3}$

$A(y)=\pi R^2=\pi/3$

$\frac{\pi}{3}\frac{dy}{dt}=-\pi(0.01)^2[(2)(9.8)y]^{1/2}$

$2\sqrt{y}=-3(0.01)^2[(2)(9.8)]^{1/2}t+C$

$y(0)=1\Rightarrow C=2$

$2\sqrt{y}=-3(0.01)^2[(2)(9.8)]^{1/2}t+2$

$y=0\Rightarrow t\simeq 1505.8s$

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8)a) More accurate soln (in cm.)

$\pi y^2\frac{dy}{dt}=-\pi(1cm)^2\sqrt{2g(y-1)}$

$\Rightarrow\frac{2}{15}\sqrt{y-1}(8+4y+3y^2)=-\sqrt{2g}t+C$

Solve for $y=100$ cm at $t=0$

we get $C=40340.77$

and for $y=1$ cm we get $t=911.2$ s

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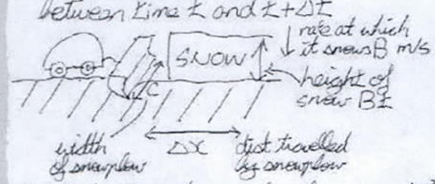
9)a) Consider what happened between time t and $t+\Delta t$



Let D be the rate snowplow clears snow (m^3/s)

Amount of snow cleared by snowplow between t and $t+\Delta t$ $=D\Delta t=(C)(\Delta x)(Bt)$

$\Rightarrow\frac{\Delta x}{\Delta t}=\frac{D}{B\times C}\times\frac{1}{t}$

As $\Delta t\to 0$, $k\frac{dx}{dt}=\frac{1}{t}$ where $k=\frac{BC}{D}$

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9)b) Suppose 7AM is $t=\tau$

$\int_0^2 k\,dx=\int_\tau^{\tau+1}\frac{1}{t}dt\Rightarrow 2k=\ln\frac{\tau+1}{\tau}$

$\int_2^4 k\,dx=\int_{\tau+1}^{\tau+3}\frac{1}{t}dt\Rightarrow 2k=\ln\frac{\tau+3}{\tau+1}$

$\Rightarrow\tau=1$ So it starts snowing at 6AM

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10)a) (mass of rocket) × (accel. of rocket) = (rate at which fuels are expelled) (velocity of fuels rel. to rocket)

$m(t)\frac{dv(t)}{dt}=-u(t)\frac{dm(t)}{dt}$

10)b) $\frac{dm}{dt}+\frac{1}{u_0}\frac{dv}{dt}m=0$

$I=\exp(\int\frac{1}{u_0}\frac{dv}{dt}dt)=\exp(\frac{v}{u_0})$

$\frac{d}{dt}e^{\frac{v}{u_0}}m=0$

$e^{v/u_0}m=C_0$

$e^{v(0)/u_0}m(0)=C_0$

$e^{v(t)/u_0}m(t)=e^{v(0)/u_0}m(0)$

$\Rightarrow v(t)=v(0)+u_0\ln\left[\frac{m(0)}{m(t)}\right]$

18.03 Differential Equation - Fall 2004

Problem Set Two

Due by noon on Friday, September 24.

This homework is graded out of 100 marks. Each question carries 10 marks.

Collaboration is encouraged in this course. If you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. You must turn in your own writeups of all problems, and if you do collaborate, please write on your solution sheet the names of the students you worked with.

1a. Find the general solution of $\frac{dy}{dx} = y(xy^3 - 1)$

1b. Find the general solution of $x\frac{dy}{dx} - (1+x)y = xy^2$

2a. Find the general solution of $x\frac{dy}{dx} = y + x\exp\left(\frac{y}{x}\right)$

2b. Find the general solution of $\frac{dy}{dx} = \frac{x^2-y^2}{5xy}$

3. Solve the differential equation $\frac{dy}{dx} = \frac{x-y-1}{x+y+3}$ by finding h and k so that the substitution $x = u + h$, $y = v + k$ transform it into the homogeneous equation $\frac{dv}{du} = \frac{u-v}{u+v}$

4. Suppose that the population $P(t)$ of a country satisfies the differential equation $\frac{dP}{dt} = kP(200 - P)$ with constant k. Its population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

5a. Draw a few isoclines for the ODE $y' = 2y + x$.

5b. Use the isoclines drawn in (a) to sketch the direction field of the equation.

5c. Mark on the xy-plane the solution curve with initial condition $y(0) = 2$.

6a. Either use dfield6 or by hand, draw the direction field of the ODE $\frac{dy}{dx} = x^2 - y - 1$ for $-3 \le x, y \le 3$.

6b. Indicate on your plot the two solution curves that passes through $(0, -2)$ and $(0, 2)$.

7. Consider the ODE $y' = y(y - x)$ with initial condition $y(0) = y_0$. Solutions with $y_0 > y_{bif}$ seem to rise toward ∞ while those with $y_0 < y_{bif}$ rise a bit but fall toward 0. The value of y_{bif} is in range $0.790 < y_{bif} < 0.800$. Note that y_{bif} is the value of y_0 at the bifurcation point between rising and (eventually) falling solutions. In part b, c, you should express the solution in terms of a function defined by an integral.

7a. Use dfield6 to find y_{bif} to 4 significant figures.

7b. Solve the Bernoulli equation analytically and find the exact value of y_{bif}. Show all your reasoning. (Hint: $\int_0^\infty e^{-x^2/2}dx = \sqrt{(\pi/2)}$.

7c. Identify the separatrix, the curve which separates the xy-plane into regions, where solutions have the two distinct behaviors discussed above.

8. Suppose that a deer population $x(t)$ (in kilodeer) is modeled by the logistic equation $\frac{dx}{dt} = x(1 - x^2) = g(x)$.

8a. Sketch the function $g(x)$. (Allow both positive and negative x)

8b. Draw the phase line and sketch a few solutions of each type.

8c. Hunters are introduced into this system. They harvest the deer at a constant rate of h kilodeer per unit time. Describe what happens to the phase line as h increases from zero. (There is a critical value of h above which the population collapses to zero.)

9. Derive the solution $P(t) = \frac{MP_0}{P_0+(M-P_0)\exp(-kMt)}$ of the logistic initial value problem $P' = kP(M - P)$, $P(0) = P_0$. Make it clear how your derivation depends on whether $0 < P_0 < M$ or $P_0 > M$.

10. Consider the first order autonomous ODE $x'(t) = (x - 10)(x + 10)^4(x + 20)$.

a. Find the critical points for the ODE.

b. Determine and sketch the phase diagram for this equation.

c. Sketch a few representative solution curves for this ODE (Exact solution is unnecessary).

DIRECTION FOR dfield6 ON ATHENA:

add matlab

matlab

(Matlab 7.0 Command window will open after a while)

>>DFIELD6

(Setup window appears)

Enter your ODE and set names and ranges of variables. (It seems that you must use t as independent variable.)

Click "Proceed" to get direction field in DFIELD6 Display Window.

Click on any point in Display to produce solution curve.

Click on options and test out "Keyboard input, Plot level curves, Plot several solutions" (caution on pss).

18.03 SOLUTION SET 2

(1a)

$$\begin{aligned} y' &= y(xy^3 - 1) \\ y' + y &= xy^4 \end{aligned}$$

This is a Bernoulli equation with $n = 4$. Let $v = y^{1-n} = y^{-3}$. This gives a new differential equation

$$v' - 3v = -3x.$$

The integrating factor is

$$I(x) = e^{\int -3dx} = e^{-3x},$$

which transforms the equation into

$$\begin{aligned} \frac{d}{dx}(e^{-3x}v) &= -3xe^{-3x} \\ e^{-3x}v &= xe^{-3x} + \frac{1}{3}e^{-3x} + C \\ v &= x + \frac{1}{3} + Ce^{3x}. \end{aligned}$$

This gives

$$y^{-3} = x + \frac{1}{3} + Ce^{3x}.$$

(1b)

$$xy' - (1+x)y = xy^2 \Rightarrow y' - (\tfrac{1}{x} + 1)y = y^2.$$

This is a Bernoulli equation with $n = 2$, so we let $v = y^{1-2} = y^{-1}$. This gives a new differential equation

$$v' + (1 + \tfrac{1}{x})v = -1.$$

The integrating factor is

$$I(x) = e^{\int 1 + \frac{1}{x} dx} = xe^x,$$

so our differential equation becomes

$$\begin{aligned} \frac{d}{dx}(xe^x v) &= -xe^x \Rightarrow xe^x v = -xe^x + e^x + C \\ v &= -1 + \tfrac{1}{x} + \tfrac{C}{x}e^{-x}. \end{aligned}$$

This gives

$$y^{-1} = -1 + \tfrac{1}{x} + \tfrac{C}{x}e^{-x}.$$

(2a)

$$xy' = y + xe^{\frac{y}{x}} \Rightarrow y' = \tfrac{y}{x} + e^{\frac{y}{x}}.$$

(4) We start by recalling that everything is measured in millions of people. If we start with $t = 0$ at 1940, then the initial conditions mean that $P(0) = 100$ and $P'(0) = 1$. Plugging this in gives

$$1 = P'(0) = kP(0)(200 - P(0)) = (100)^2 k \Rightarrow k = (100)^{-2}.$$

Now we solve

$$\begin{aligned} \frac{100^2}{P(200-P)}dP &= dt \Rightarrow 100\left(\frac{1}{2}\left(\frac{1}{p} + \frac{1}{200-P}\right)\right)dP = dt \\ 50(\ln(P) - \ln(200-P)) &= t + C \Rightarrow \frac{P}{200-P} = Ae^{t/50}. \end{aligned}$$

The condition $P(0) = 100$ gives us

$$\frac{100}{200-100} = 1 = A.$$

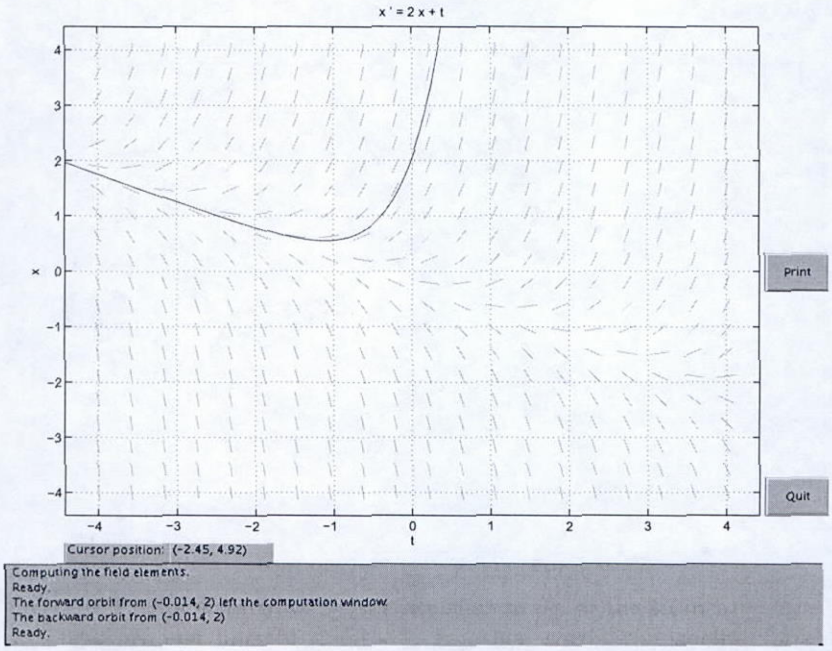
Solving the last equation for P gives us

$$P = 200e^{t/50} - e^{t/50}P \Rightarrow P = \frac{200e^{t/50}}{1 + e^{t/50}}.$$

This gives the population in the year 2000 as

$$P(60) = \frac{200e^{6/5}}{1 + e^{6/5}} \approx 153.7.$$

(5a) The isoclines for $y' = 2y + x$ are the curves of the form $m = 2y + x$ for m fixed. These are just lines of slope $-\frac{1}{2}$. In other words, the isocline for slope m is the line $y = -\frac{1}{2}x + \frac{m}{2}$. Using this we can get the slope field:



In the picture, the solution curve through $(0, 2)$ is the indicated curved line.

(7b) If we solve analytically, we get the following:

$$y' = y^2 - xy \Rightarrow y' + xy = y^2.$$

This is a Bernoulli equation with $n = 2$, so we let $v = y^{1-n}$. This gives a new ODE

$$v' - xv = -1.$$

The integrating factor for this is just

$$I(x) = e^{\int_0^x -x dx} = e^{-x^2/2}.$$

This transforms our equation into

$$\begin{aligned}
\frac{d}{dx}(ve^{-x^2/2}) &= -e^{-x^2/2} \Rightarrow ve^{-x^2/2} - v(0) = \int_0^x -e^{-x^2/2}dx \\
v(x) &= -e^{x^2/2}\int_0^x e^{-x^2/2}dx + v(0)e^{x^2/2}
\end{aligned}$$

If we recall that $v = y^{-1}$, then we see that $v(0) = y(0)^{-1}$, and this gives

$$y(x) = \frac{y(0)}{e^{x^2/2} - y(0)e^{x^2/2}\int_0^x e^{-x^2/2}dx} = \frac{y(0)e^{-x^2/2}}{1 - y(0)\int_0^x e^{-x^2/2}dx}.$$

The solutions will blow up if the denominator every vanishes. This means that there would be some x such that $1 - y(0)\int_0^x e^{-x^2/2}dx = 0$ This just means that

$$y(0) > \frac{1}{\int_0^\infty e^{-x^2/2}dx} = \sqrt{\frac{2}{\pi}},$$

since the integral is a strictly increasing function going from 0 to the (never achieved) maximum of $\int_0^\infty e^{-x^2/2}dx = \sqrt{\pi/2}$. If $y(0)$ is less than this critical value, then the denominator never vanishes and the numerator eventually forces the value of the entire function down towards 0. We conclude that

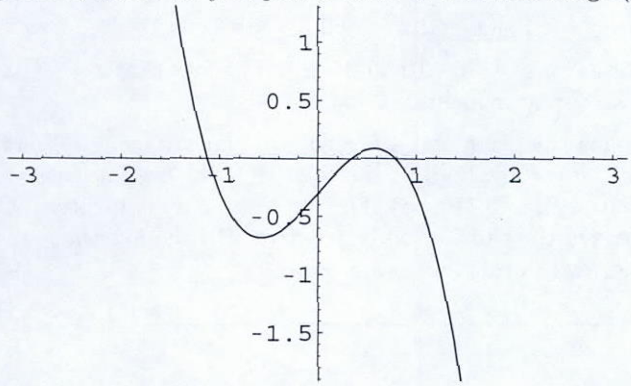
$$y_{bif} = \sqrt{\frac{2}{\pi}} \approx 0.7978.$$

(7c) The separatrix is the curve that occurs when $y(0) = y_{bif}$. This is just the curve

$$y(x) = \frac{\sqrt{\frac{2}{\pi}}e^{-x^2/2}}{1 - \sqrt{\frac{2}{\pi}}\int_0^x e^{-x^2/2}dx}.$$

(8a) Here is a sketch of the curve $g(x)$.

where h is the constant rate at which hunters kill deer. As h increases from 0 to $g(x_0)$, the top to critical points get closer and closer together. We can see this from a picture, where we just pick a value of h in this range (here $h = .3$):



When $h = g(x_0)$, $g(x) - h$ stops having 3 solutions and only has 2. At the bigger critical point, the phase diagram shows that there are no positive values of x at which the population is increasing. In other words, at this point, the population continuously decays to 0. Beyond the value $h = g(x_0)$, the function $g(x) - h$ only has one solution, and for all positive values of x, the solution decreases to 0. The phase diagram here has a single critical point corresponding to a negative x value and which is stable.

(9) We have $\frac{dP}{dt} = kP(M - P)$. The critical points are $P = 0$ and $P = M$, and if we draw a phase diagram, then we see that M is stable whereas 0 is not. This will just help us ensure that everything makes sense in our solutions.

We solve by separation of variables:

$$\begin{aligned} \frac{dP}{P(M-P)} &= \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = kdt \\ \ln(P) - \ln|M-P| &= Mkt + C \Rightarrow \frac{P}{|M-P|} = Ae^{Mkt} \end{aligned}$$

If we recall that $P(0) = P_0$, then we see that $A = \frac{P_0}{|M-P_0|}$. Now we have two cases

If $P_0 < M$, then $|M - P_0| = M - P_0$. Since M is a stable critical value, we know that P starts less than M and always stays there, so $|M - P| = M - P$. Plugging everything in and solving, we get

$$\begin{aligned} \frac{P}{M-P} &= \frac{P_0}{M-P_0}e^{Mkt} \Rightarrow (M-P_0)P = P_0(M-P)e^{Mkt} \\ ((M-P_0)e^{-Mkt} + P_0)P &= MP_0 \rightarrow P = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}. \end{aligned}$$

If $P_0 > M$, then $|M - P_0| = P_0 - M$, and we know that since P starts bigger than M, we conclude that it is always at least M, and $|M - P| = P - M$. Plugging everything in and solving, we get

$$\begin{aligned} \frac{P}{P-M} &= \frac{P_0}{P_0-M}e^{Mkt} \Rightarrow (P_0-M)P = P_0(P-M)e^{Mkt} \\ ((P_0-M)e^{-Mkt} - P_0)P &= -MP_0 \rightarrow P = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}. \end{aligned}$$

Engineering Mathematics I - Spring 2005

Problem Set 1

1 a Find the general solution of

$$y^3 \frac{dy}{dx} = (y^4 + 3)\cos x$$

1 b Find the general solution of

$$\frac{dy}{dx} \tan x = y$$

2 a Solve the initial value problem

$$\frac{dy}{dx} = \frac{4x - y}{x - 6y}, \qquad y(1) = 1$$

2 b Solve the initial value problem

$$(x^2 + 1)\frac{dy}{dx} + 3x^3 y = 6x \exp(-\frac{3}{2}x^2), \qquad y(0) = 5$$

3 Find the general solution of

$$\frac{dy}{dx} + 5xy = \frac{y \ln y}{x}$$

by substituting $v = \ln y$

4 Find the general solution of

$$\frac{dy}{dx} = x^2 + y^2 + 2xy + 4x + 4y + 4$$

5 a Find the general solution of

$$y \frac{dx}{dy} + (2y - 3)x = 4y^4$$

(Treat y as independent variable and find $x(y)$)

5 b Solve the initial value problem

$$x \frac{dy}{dx} + 3y = 2x^5, \qquad y(2) = 3$$

6 a Find the general solution of

$$\frac{dy}{dx} = y(xy^3 - 1)$$

6 b Find the general solution of

$$x \frac{dy}{dx} - (1 + x)y = xy^2$$

7 a Find the general solution of

$$x\frac{dy}{dx} = y + x\exp(\frac{y}{x})$$

7 b Find the general solution of

$$\frac{dy}{dx} = \frac{x^2 - y^2}{5xy}$$

8 A tank contains 1000 liters (L) of a solution consisting of 50 kg salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture - kept uniform by stirring - is pumped out at the same rate. How long will it be until only 3 kg of salt remains in the tank?

9 Suppose that you discover in your backpack an overdue engineering mathematics textbook on which your friend owed a fine of 5 won 100 days ago. If an overdue fine grows exponentially at a 5% daily rate compounded continuously, how much would you have to pay if you return the book today?

10 Suppose that the population $P(t)$ of a country satisfies the differential equation $\frac{dP}{dt} = kP(200-P)$ with constant k. Its population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

Engineering Mathematics I - Spring 2005

Sample Midterm Test

April 21, 2005

1 Solve the following differential equations. An implicit solution is acceptable.

a

$$(x+y)y' = y$$

b

$$\frac{dy}{dx} = \frac{2\ln y + x}{\sin y - 2x/y}$$

2 Find the solution to the initial value problem

$$\frac{dy}{dt} + y = t \qquad y(0) = 0$$

3

a Solve the following:

$$\frac{dy}{dx} = \frac{1 - 3x^2 - y^2}{2xy}$$

b Find the solution curve of this ODE that passes through the point (1,0). Sketch the solution curve on the x-y plane. What is the geometrical shape of this curve?

4 Solve the initial value problem

$$y'' + 4y' + 5y = 0 \qquad y(0) = 2, \quad y'(0) = 0$$

5 Write next to each of the following ODE's

(1) $y'' + y' - 6y = 0$

(2) $y'' + 5y' + 6y = 0$

(3) $y'' + 5y' + 6y = t^2$

(4) $y'' - 2y' + 5y = e^{-t}$

the letter which best describes the solutions:

A. All solutions tend to 0 as $t \to +\infty$.

B. At least one solution tends to 0 as $t \to +\infty$ but not all solutions have this property.

C. None of the above.

6 Knowing that $x/2 + x^2$ and $x/2 - 3x^3$ are two particular solutions of

$$x^2y'' + P(x)y' + 6y = x,$$

find $P(x)$. What is the homogeneous solution to this equation?

7 Find the general solution of $x^2y'' + 4xy' - 4y = x^2$.

8 Solve the initial problem $y'' + \pi^2 y = f(t)$ for the source

$$f(t) = \begin{cases} \pi^2 t & \text{for } 0 \le t \le 1 \\ \pi^2(2-t) & \text{for } 1 \le t \le 2 \\ 0 & \text{for } t > 2 \end{cases}$$

with initial conditions $y(0) = 0$, $y'(0) = 0$.

9 Find the general solution of the system

$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} e^{-t} \\ e^{t} \end{pmatrix}$$

10 Consider the nonlinear system:

$$\frac{dx}{dt} = x^2 - y$$

$$\frac{dy}{dt} = (x-1)^2 + y - 1$$

a Find all critical points of the system.

b At each critical point, linearize the system and indicate the type of the critical point.

c Sketch the phase portrait of the system in the xy-plane.

Sample Midterm Test.

1.

(a) $(x+y)\,y' = y$

$$\frac{dy}{dx} = \frac{y}{x+y} = \frac{(y/x)}{1+(y/x)}$$

let $u = \frac{y}{x}$. $\quad y = ux$

$$y' = u'x + u$$

$$u'x + u = \frac{u}{1+u}$$

$$xu' = \frac{u}{1+u} - u = \frac{u-u^2-u}{1+u} = -\frac{u^2}{1+u}$$

$$x\frac{du}{dx} = -\frac{u^2}{1+u}$$

$$\frac{1+u}{u^2}\,du = -\frac{dx}{x}$$

$$(u^{-2}+u^{-1})\,du = -\frac{dx}{x}$$

$$-\frac{1}{u} + \ln u = -\ln x + c$$

$$-\frac{x}{y} + \ln\frac{y}{x} + \ln x = c$$

$$-\frac{x}{y} + \ln y = c$$

(b) $$\frac{dy}{dx} = \frac{2\ln y + x}{\sin y - 2\frac{x}{y}}$$

$$\underbrace{(2\ln y + x)}_{=M}\,dx \underbrace{-\left(\sin y - 2\frac{x}{y}\right)}_{=N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2}{y}$$

$$\frac{\partial N}{\partial x} = \frac{2}{y} \qquad \therefore \text{ exact}$$

$$\frac{\partial u}{\partial x} = 2\ln y + x$$

$$u = 2x\ln y + \tfrac{1}{2}x^2 + f(y)$$

$$\frac{\partial u}{\partial y} = 2x\frac{1}{y} + f'(y) = -\sin y + \frac{2x}{y}$$

$$f(y) = \cos y + c$$

$$2x\ln y + \frac{x^2}{2} + \cos y = c.$$

3 (a) $\dfrac{dy}{dx} = \dfrac{1-3x^2-y^2}{2xy}$

$$\underbrace{(3x^2+y^2-1)}_{=M} dx + \underbrace{2xy}_{=N} dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y.$$

$$\frac{\partial u}{\partial x} = 3x^2+y^2-1. \qquad \frac{\partial u}{\partial y} = 2xy$$

$$u = xy^2 + f(x)$$

$$\frac{\partial u}{\partial x} = y^2 + f' = 3x^2+y^2-1. \qquad f' = 3x^2-1$$

$$f = x^3 - x + c$$

$$\therefore\ xy^2 + x^3 - x = \text{const}$$

2. $\frac{dy}{dt} + y = t \qquad y(0) = 0.$

$dy + (y-t)dt = 0$: $f_1(t)\,dy + f_1(y-t)\,dt = 0$

$\frac{dF}{dt} = F \qquad \frac{dF}{F} = dt \qquad \ln f_1 = t.$

integ. factor $f_1 = e^t$

$e^t dy + e^t (y-t)\,dt = 0$

$\frac{\partial u}{\partial y} = e^t. \qquad u = e^t y + f(t)$

$\frac{\partial u}{\partial t} = e^t y + f' = e^t y - te^t.$

$f' = -te^t$

$f = -\int \underbrace{t}_{u} \underbrace{e^t dt}_{v'} = -\left[te^t - \int e^t dt\right]$

$= -te^t + e^t \overset{+c}{} = -(+t-1)e^t + c$

$u = e^t \cdot y + (-t+1)e^t + c = \text{const.}$

$ye^t = (t-1)e^t + c$

$y = t-1 + ce^{-t}$

$y(0) = -1 + c = 0 \qquad c = 1$

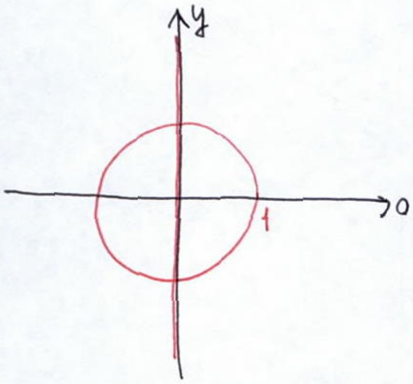
$\therefore y = t-1+e^{-t}$

3 (b) $(1,0) \Rightarrow \quad 1\cdot 0 + 1 - 1 = 0.$

$xy^2 + x^3 - x = 0$

$x(y^2 + x^2 - 1) = 0$

$x = 0 \quad \text{or} \quad x^2 + y^2 = 1$



y-axis & unit circle

4 $y'' + 4y' + 5y = 0$

$y = e^{\lambda x}$

$\lambda^2 + 4\lambda + 5 = 0 \qquad \lambda = -2 \pm \sqrt{4-5} = -2 \pm i$

$y = e^{-2x}(c_1 \cos x + c_2 \sin x)$

$y(0) = c_1 = 2$

$y' = -2e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-2x}(-c_1 \sin x + c_2 \cos x)$

$y'(0) = -2c_1 + c_2 = 0. \qquad c_2 = 4$

$\therefore \; y = e^{-2x}(2\cos x + 4\sin x)$

5. (1) $y''+y'-6y=0$

$\lambda^2+\lambda-6=0.$ $(\lambda+3)(\lambda-2)=0.$ $y=c_1e^{-3t}+c_2e^{2t}$: B

(2) $y''+5y+6y=0$

$\lambda^2+5\lambda+6=0.$ $(\lambda+2)(\lambda+3)=0$ $y=c_1e^{-2t}+c_2e^{-3t}$: A

(3) $y''+5y'+6y=t^2$

$y_p=at^2+bt+c$

$y_p'=2at+b$

$y_p''=2a$

$2a+10at+5b+6at^2+6bt+6c=t^2$

$a=\frac{1}{6}.$ $10a+6b=0.$

$b=-\frac{10}{6}a=-\frac{5}{3}\left(\frac{1}{6}\right)=-\frac{5}{18}.$

$2a+5b+6c=0$

$\frac{1}{3}-\frac{25}{18}+6c=0.$ $6c=\frac{25-6}{18}=\frac{19}{18}$

$c=\frac{19}{108}$

$y_p=\frac{1}{6}t^2-\frac{5}{18}t+\frac{19}{108}$

$y=c_1e^{-2t}+c_2e^{-3t}+\frac{1}{6}t^2-\frac{5}{18}t+\frac{19}{108}$: B

(4) $y''-2y'+5y=e^{-t}$

y_h: $\lambda^2-2\lambda+5=0$ $\lambda=1\pm\sqrt{1-5}=1\pm2i$

-1

$y_p=ae^{-t}.$ $y_p'=-ae^{-t}.$ $y_p''=ae^{-t}$

$(a-2a+5a)e^{-t}=4ae^{-t}=e^{-t}.$ $a=\frac{1}{4}$

$y = e^{t}(c_1\cos 2t + c_2\sin 2t) + \frac{1}{4}e^{-t}$: B

6. $x^2y'' + p(x)y' + 6y = x.$

$y_{p1} = \frac{x}{2} + x^2.$ $\frac{x}{2} - 3x^3$

$y_{p1}' = \frac{1}{2} + 2x$

$y_{p1}'' = 2$

$2x^2 + p(x)(\frac{1}{2} + 2x) + 3x + 6x^2 = x$

$(\frac{1}{2} + 2x)\,p(x) = -8x^2 - 2x$

$$p(x) = \frac{-x(8x+2)}{2x+\frac{1}{2}} = \frac{-2x(4x+1)}{\frac{1}{2}(4x+1)} = -4x$$

$x^2y'' - 4xy' + 6y = x$

$y_h = x^m$

$m(m-1) - 4m + 6 = 0$

$m^2 - 5m + 6 = 0$ $(m-2)(m-3) = 0$

$y_h = c_1x^2 + c_2x^3$

$\therefore y = c_1x^2 + c_2x^3 + \frac{x}{2}$

7. $x^2y'' + 4xy' - 4y = x^2$

y_h : x^m

$m(m-1) + 4m - 4 = m^2 + 3m - 4 = (m+4)(m-1) = 0$

$y_h = c_1 x^{-4} + c_2 x$

$y_p = ax^2 + \cancel{bx} + \cancel{c}$

$y_p' = 2ax$

$y_p'' = 2a$

$2ax^2 + 8ax^2 - 4ax^2 - \cancel{4c} = x^2$

$6ax^2 = x^2$. $a = \frac{1}{6}$.

$\therefore y = c_1 x^{-4} + c_2 x + \frac{1}{6}x^2$

8. $0 \leq t \leq 1$

$y'' + \pi^2 y = \pi^2 t$

y_h: $\lambda^2 + \pi^2 = 0$. $\lambda = \pm \pi i$. $y_p = t$.

$\therefore y = c_1 \cos\pi t + c_2 \sin\pi t + t$

$y(0) = c_1 = 0$

$y' = -\pi \cancel{c_1} \sin\pi t + \pi c_2 \cos \pi t + 1 = 0$

$\pi c_2 = -1$. $c_2 = -\frac{1}{\pi}$

$y = -\frac{1}{\pi}\sin\pi t + t \quad (0 \leq t \leq 1)$.

$1 \leq t \leq 2$

$y'' + \pi^2 y = \pi^2(2-t)$

$y_p = 2-t$. $y = c_1 \cos\pi t + c_2 \sin\pi t + 2 - t$.

$y(1) = -c_1 + 2 - 1 = 1$. $\quad$ previous y $\quad c_1 = 0$

$y' = \pi c_2 \cos \pi t - 1$. $\quad$ prev $\quad y' = -\cos \pi t + 1$

$y'(1) = -\pi c_2 - 1 = 2$. $\quad c_2 = -\frac{3}{\pi}$

$$\therefore y = -\frac{3}{\pi} \sin \pi t + 2 - t \qquad (1 \le t \le 2)$$

$$y' = -3 \cos \pi t - 1$$

$t > 2$

$y'' + \pi^2 y = 0$

$y = c_1 \cos \pi t + c_2 \sin \pi t$. $\qquad y' = -c_1 \pi \sin \pi t + c_2 \pi \cos \pi t$

$y(2) = c_1 = 0$.

$y'(2) = \pi c_2 = -4$. $\quad c_2 = -\frac{4}{\pi}$

$$\therefore y = -\frac{4}{\pi} \sin \pi t \qquad (t > 2).$$

9. $\quad \bar{y}' = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \bar{y} + \begin{bmatrix} e^{-t} \\ e^{t} \end{bmatrix}$

$\bar{y}^{(h)} = \bar{x} e^{\lambda t}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda+3)(\lambda-3) + 8 = \lambda^2 - 1 = 0$$

$$\lambda = 1, -1$$

$$(A - \lambda I)\bar{x} = \begin{bmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$\lambda_1 = 1$: $\quad 2x_1 - 4x_2 = 0 \quad x_1 = 2x_2. \quad \bar{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\qquad\qquad 2x_1 - 4x_2 = 0.$

$\lambda_2 = -1$: $\quad 4x_1 - 4x_2 = 0 \quad \bar{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\qquad\qquad 2x_1 - 2x_2 = 0$

$$\bar{y}^{(h)} = c_1 \begin{bmatrix}2\\1\end{bmatrix} e^{t} + c_2 \begin{bmatrix}1\\1\end{bmatrix} e^{-t}$$

$$\bar{y}_p = (\bar{u}t + \bar{v})e^{t} + (\bar{w}t + \bar{z})e^{-t}$$

$$\bar{y}_p' = \bar{u}e^{t} + (\bar{u}t + \bar{v})e^{t} + (\bar{w} - \bar{w}t - \bar{z})e^{-t}$$

$$(\bar{u}+\bar{v})e^{t} + \bar{u}te^{t} + (\bar{w}-\bar{z})e^{-t} - \bar{w}te^{-t}$$

$$= A\left[(\bar{u}t+\bar{v})e^{t} + (\bar{w}t+\bar{z})e^{-t}\right] + \overbrace{\begin{bmatrix}e^{-t}\\e^{t}\end{bmatrix}}^{/\!/\ \begin{bmatrix}1\\0\end{bmatrix}e^{-t} + \begin{bmatrix}0\\1\end{bmatrix}e^{t}}$$

e^{t}: $\bar{u}+\bar{v} = A\bar{v} + \begin{bmatrix}0\\1\end{bmatrix}$: $(A-I)\bar{v} = \bar{u} - \begin{bmatrix}0\\1\end{bmatrix}$ $\cdots$ ①

te^{t}: $\bar{u} = A\bar{u}$: $\bar{u} = a\begin{bmatrix}2\\1\end{bmatrix}$

e^{-t}: $\bar{w}-\bar{z} = A\bar{z} + \begin{bmatrix}1\\0\end{bmatrix}$: $(A+I)\bar{z} = \bar{w} - \begin{bmatrix}1\\0\end{bmatrix}$ $\cdots$ ②

te^{-t}: $-\bar{w} = A\bar{w}$: $\bar{w} = b\begin{bmatrix}1\\1\end{bmatrix}$

① $\Rightarrow$ $\begin{bmatrix}2 & -4\\2 & -4\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}2v_1-4v_2\\2v_1-4v_2\end{bmatrix} = \begin{bmatrix}2a\\a-1\end{bmatrix}$

$2a = a-1.$ $a=-1.$

$2v_1 - 4v_2 = -2$ $v_1 - 2v_2 = -1.$ $\cancel{\bar{v} = \begin{bmatrix}1\\0\end{bmatrix}}$

$v_2 = k$
$v_1 = 2k-1$ $\bar{v} = \begin{bmatrix}2k-1\\k\end{bmatrix}$

$= k\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix}$

② $\Rightarrow$ $\begin{bmatrix}4 & -4\\2 & -2\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}b-1\\b\end{bmatrix}.$ $4(z_1-z_2) = b-1$
$2(z_1-z_2) = b$

$2 = \frac{b-1}{b}.$ $2b = b-1$ $b=-1.$ $z_1 - z_2 = -1/2$

$z_1 = l.$ $z_2 = l + 1/2$

$\bar{z} = \begin{bmatrix}l\\l+1/2\end{bmatrix} = l\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\1/2\end{bmatrix}$

$$\bar{y}_p = \left\{-\begin{bmatrix}2\\1\end{bmatrix}t + \begin{bmatrix}-1\\0\end{bmatrix}\right\}e^t + \left\{-\begin{bmatrix}1\\1\end{bmatrix}t + \begin{bmatrix}0\\1/2\end{bmatrix}\right\}e^{-t}$$

$$= \begin{bmatrix}(-2t-1)e^t + (-t)e^{-t} \\ -te^t + (-t+\frac{1}{2})e^{-t}\end{bmatrix}$$

$$\bar{y} = c_1\begin{bmatrix}2\\1\end{bmatrix}e^t + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-t} + \bar{y}_p$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}(2c_1-1)e^t - 2te^t + c_2e^{-t} - te^{-t} \\ c_1e^t \quad - te^t + (c_2 + \frac{1}{2})e^{-t} - te^{-t}\end{bmatrix}$$

10. $\begin{cases} x' = x^2 - y \\ y' = (x-1)^2 + y - 1 \end{cases}$

(a) $x' = 0. \quad x^2 = y$

$y' = 0. \quad (x-1)^2 + x^2 - 1 = 0$

$2x^2 - 2x = 2x(x-1) = 0$

$\begin{cases} x=0. \quad y=0 \\ x=1. \quad y=1 \end{cases}$

crit. pts $(0,0)$, $(1,1)$

(b) At $(0,0)$ $\quad x, y.$: small

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-y\\-2x+y\end{bmatrix} = \begin{bmatrix}0 & -1\\-2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

$$\det(A-\lambda I) = \begin{vmatrix}-\lambda & -1\\-2 & 1-\lambda\end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$$

$\lambda = -1, 2$ ∴ saddle

At $(1,1)$. $\quad x-1 = \tilde{x}, \quad y-1 = \tilde{y}$

$$\tilde{x}' = (\tilde{x}+1)^2 - (\tilde{y}+1) \cong 2\tilde{x} - \tilde{y}$$

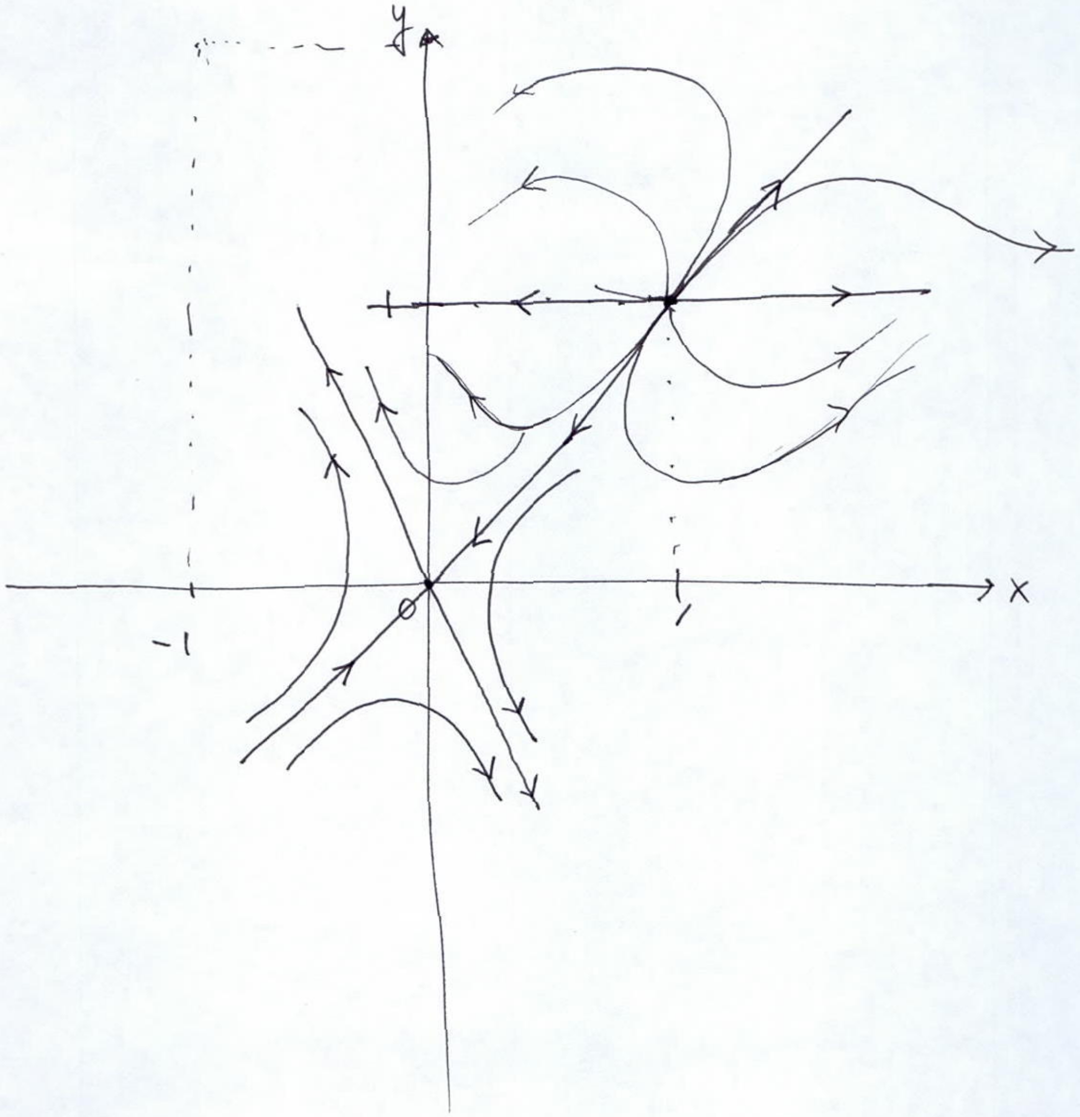
$\tilde{y}' = \tilde{y}$

$$\begin{bmatrix}\tilde{x}'\\ \tilde{y}'\end{bmatrix} = \begin{bmatrix}2 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix}2-\lambda & -1\\ 0 & 1-\lambda\end{vmatrix} = (\lambda-1)(\lambda-2) = 0. \qquad \lambda = +1,\ +2$$

improper node

(c)



$(0,0)$: $\begin{bmatrix}-\lambda & -1\\ -2 & 1-\lambda\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0.$ $\quad \lambda_1 = -1$: $v_1 - v_2 = 0$ $\quad \vec{v}^{(1)} = \begin{bmatrix}1\\ 1\end{bmatrix}$

$\lambda_2 = 2$: $-2v_1 - v_2 = 0$ $\quad \vec{v}^{(2)} = \begin{bmatrix}1\\ -2\end{bmatrix}.$

$v_2 = -2v_1$

$$\vec{y} = c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-t} + c_2\begin{bmatrix}1\\ -2\end{bmatrix}e^{2t}$$

$(1,1)$: $\begin{bmatrix}2-\lambda & -1\\ 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = 0.$ $\quad \lambda_1 = 1$: $u_1 - u_2 = 0$ $\quad \vec{u}^{(1)} = \begin{bmatrix}1\\ 1\end{bmatrix}$ $\quad \lambda_2 = 2$: $0\cdot u_1 - u_2 = 0.$ $\quad 0\cdot u_1 - u_2 = 0.$ $\quad \vec{u}^{(2)} = \begin{bmatrix}1\\ 0\end{bmatrix}.$

$$\tilde{\vec{y}} = c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{t} + c_2\begin{bmatrix}1\\ 0\end{bmatrix}e^{2t}$$

Engineering Mathematics I - Spring 2005

Midterm Examination

April 25, 2005

1

1a (10 points) Solve the initial value problem

$$\frac{dy}{dx} + \frac{2y}{x} = x^3, \qquad y(1) = 1$$

1b (10 points) Find the general solution of

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

1c (10 points) Find the general solution of

$$\frac{dy}{dx} + 2xy = 1 + x^2 + y^2$$

by using the substitution $y = x + \frac{1}{v}$. *Hint*: Remember that v is a function of x.

2

2a (10 points) Solve the initial value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 12y = 0, \qquad y(0) = y'(0) = 1$$

2b (10 points) Find the general solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}$$

2c (10 points) Consider the ordinary differential equation $y'' + 2y' + ky = 0$. For what values of k will the solutions be oscillatory, i.e. of the form $e^{at}\cos\omega t$, $e^{at}\sin\omega t$?

2d (10 points) A differential equation of the form $y'' + p(x)y' + q(x)y = 0$ is known to have $y_1(x) = x$ and $y_2(x) = e^x$ as solutions. Find $p(x)$ and $q(x)$.

3 For the nonlinear system

$$\frac{dx}{dt} = x - y$$

$$\frac{dy}{dt} = -y + 2x^2 - xy$$

3a (10 points) find the two critical points.

3b (10 points) At each critical point, linearize the system and indicate the *type* and *stability* of the critical point.

3c (10 points) Sketch the phase portrait around the two critical points in the xy-plane.

Midterm Exam Solutions

1

(a) $x\frac{dy}{dx}+2y=x^4$

$x\,dy=(x^4-2y)\,dx$

$\underbrace{(x^4-2y)}_{\overset{\shortparallel}{P}}dx\ \underbrace{-x}_{\overset{\shortparallel}{Q}}\,dy=0$

$\frac{\partial P}{\partial y}=-2 \qquad \frac{\partial Q}{\partial x}=-1 \;\therefore\;$ not exact

$FP\,dx+FQ\,dy=0. \quad F(x)$: integration factor

$\frac{\partial(FP)}{\partial y}=\frac{\partial(FQ)}{\partial x}$

$F\frac{\partial P}{\partial y}=\frac{dF}{dx}Q+F\frac{\partial Q}{\partial x}$

$\frac{dF}{dx}Q=F\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$\frac{dF}{dx}(-x)=F(-2+1)=-F$

$\frac{dF}{F}=\frac{dx}{x}.$

$\ln F=\ln x \qquad \therefore \quad F=x$

$(x^5-2yx)\,dx-x^2\,dy=0$

$\frac{\partial u}{\partial x}=x^5-2yx$

$\frac{\partial u}{\partial y}=-x^2 \qquad \therefore \; u=-x^2y+f(x)$

$\frac{\partial u}{\partial x}=-2xy+f'(x)=x^5-2yx$

$f(x)=\frac{1}{6}x^6+c$

$$-x^2y + \frac{1}{6}x^6 + c = \text{const}$$

$$\frac{1}{6}x^6 - x^2y = c$$

$x=1,\ y=1$

$\frac{1}{6} - 1 = c.$ $\quad c = -\frac{5}{6}$

$$\therefore\ \frac{1}{6}x^6 - x^2y = -\frac{5}{6}.$$

OR

$$x^2y = \frac{1}{6}(x^6+5)$$

$$y = \frac{1}{6}\left(x^4 + \frac{5}{x^2}\right).$$

Alternatively,

$$x^2\frac{dy}{dx} + 2xy = x^5$$

$$\frac{d}{dx}(x^2y) = x^5$$

$$x^2y = \frac{1}{6}x^6 + c$$

$(1,1) \Rightarrow\ 1 = \frac{1}{6} + c.$ $\quad c = \frac{5}{6}.$

$$y = \frac{1}{6}\left(x^4 + \frac{5}{x^2}\right)$$

1 (b)

$$\frac{dy}{dx}+2xy=e^{-x^2}$$

$$\frac{dy}{dx}=e^{-x^2}-2xy$$

$$dy=(e^{-x^2}-2xy)\,dx$$

$$\underbrace{(2xy-e^{-x^2})}_{=P}\,dx+\underbrace{dy}_{=Q}=0$$

$$\frac{\partial P}{\partial y}=2x. \qquad \frac{\partial Q}{\partial x}=0. \quad \text{not exact} \quad FP\,dx+FQ\,dy=0$$

$$\frac{dF}{dx}=F(2x-0). \qquad \frac{dF}{F}=2x\,dx$$

$$\ln F=x^2. \qquad F=e^{x^2}$$

$$(2xye^{x^2}-1)\,dx+e^{x^2}dy=0$$

$$\frac{\partial u}{\partial y}=e^{x^2}. \qquad u=e^{x^2}\cdot y+f(x)$$

$$\frac{\partial u}{\partial x}=2xye^{x^2}+ \qquad f'(x)=2xye^{x^2}-1$$

$$f'(x)= \quad -1$$

$$f(x)=-x+c$$

$$\therefore e^{x^2}\cdot y-x=c$$

$$y=e^{-x^2}(x+c)$$

Alternatively,

$$e^{x^2}\frac{dy}{dx}+2xe^{x^2}y=1$$

$$\frac{d}{dx}(e^{x^2}\cdot y)=1$$

$$e^{x^2}\cdot y=x+c$$

$$y=e^{-x^2}(x+c)$$

1 (c) $$\frac{dy}{dx}+2xy=1+x^2+y^2$$

$$y=x+\frac{1}{v}$$

$$\frac{dy}{dx}=1-v^{-2}\frac{dv}{dx}$$

$$1-\frac{1}{v^2}\frac{dv}{dx}+2x(x+\frac{1}{v})=1+x^2+(x+\frac{1}{v})^2$$

$$1-\frac{1}{v^2}\frac{dv}{dx}+2x^2+\frac{2x}{v}=1+x^2+x^2+\frac{2x}{v}+\frac{1}{v^2}$$

$$-\frac{1}{v^2}\frac{dv}{dx}=\frac{1}{v^2}$$

$$\frac{dv}{dx}=-1.$$

$$v=-x+c.$$

Recall $\frac{1}{v}=y-x$ $\quad v=\frac{1}{y-x}$

$$\frac{1}{y-x}=-x+c$$

$$(y-x)(-x+c)=1. \quad \therefore y=x+\frac{1}{c-x}$$

2 (a) $y''+4y'-12y=0.$

$y=e^{\lambda t}$

$\lambda^2+4\lambda-12=0$

$(\lambda+6)(\lambda-2)=0$

$\lambda=-6,\ 2$

$y=c_1e^{-6t}+c_2e^{2t}$

$y'=-6c_1e^{-6t}+2c_2e^{2t}$

$y(0)=c_1+c_2=1$

$y'(0)=-6c_1+2c_2=1$

$-)\ 2c_1+2c_2=2$

$-8c_1=-1.\qquad c_1=\frac{1}{8},\quad c_2=\frac{7}{8}$

$\therefore\ y=\frac{1}{8}e^{-6t}+\frac{7}{8}e^{2t}$

2 (b) $y''+3y'+2y=e^{-x}$

$y=y_h+y_p$

(1) $y_h=e^{\lambda x}$

$\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0.\qquad \lambda=-1,-2$

$y_h=c_1e^{-x}+c_2e^{-2x}$

(2) $y_p=Axe^{-x}$ (modification rule)

$y_p'=A(1-x)e^{-x}$

$y_p''=A(-e^{-x}-(1-x)e^{-x})=A(-2+x)e^{-x}$

$$[A(-2+x)+3A(1-x)+2Ax]e^{-x}=e^{-x}$$

$$(-2A+3A)+(A-3A+2A)x=1$$

$$\therefore A=1.$$

$$y_p = xe^{-x}$$

$$\therefore y = C_1e^{-x}+C_2e^{-2x}+xe^{-x}$$

2(c) $\quad y''+2y'+ky=0$

$y=e^{\lambda t}$

$\lambda^2+2\lambda+k=0$

oscillatory $\Rightarrow$ complex conjugate roots

$D=1-k<0 \qquad \therefore k>1$

2(d) $\quad y''+p(x)y'+q(x)y=0$

· $y_1=x \quad y_1'=1. \quad y_1''=0$

$0+p(x)+xq(x)=0 \qquad \therefore p=-xq$

· $y_2=e^x. \quad y_2'=e^x. \quad y_2''=e^x$

$1+p(x)+q(x)=0$

$1-xq+q = 1+(1-x)q=0$

$q=\frac{1}{x-1}. \qquad p=\frac{-x}{x-1}$

3. $\begin{cases} x' = x - y \\ y' = -y + 2x^2 - xy \end{cases}$

(a) critical pt: $x' = 0$. $y' = 0$

$x - y = 0$. $y = x$

$-y + 2x^2 - xy = 0 \Rightarrow -x + 2x^2 - x^2 = 0$

$x^2 - x = x(x-1) = 0$

$\begin{cases} x = 0, & y = 0 \\ x = 1, & y = 1 \end{cases}$

critical pt. $(0,0)$, $(1,1)$

(b) At $(0,0)$: x, y: small

$\begin{cases} x' = x - y \\ y' = -y \end{cases}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

eigenvalue: λ

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) = 0$

$\lambda_1 = -1$, $\lambda_2 = 1$: $\begin{cases} \text{type: saddle} \\ \text{stability: unstable} \end{cases}$

At $(1,1)$: $\tilde{x} = x - 1$: $x = \tilde{x} + 1$

$\tilde{y} = y - 1$: $y = \tilde{y} + 1$

$\tilde{x}' = \tilde{x} - \tilde{y}$

$\tilde{y}' = -(\tilde{y}+1) + 2(\tilde{x}+1)^2 - (\tilde{x}+1)(\tilde{y}+1)$

$= -\tilde{y} - 1 + 2(\tilde{x}^2 + 2\tilde{x} + 1) - (\tilde{x}\tilde{y} + \tilde{x} + \tilde{y} + 1)$

$= -\tilde{y} + 4\tilde{x} - \tilde{x} - \tilde{y} = 3\tilde{x} - 2\tilde{y}$

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-1)+3$$

$$= \lambda^2+\lambda+1 = 0$$

$$\lambda = \frac{1}{2}\left(-1 \pm \sqrt{1-4}\right) = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\begin{cases} \text{type : spiral} \\ \text{stability : stable} \end{cases}$$

(c) At $(0,0)$. find eigenvectors $\vec{v}$

$\lambda_1 = -1$ $\quad \begin{bmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 - v_2 \\ 0 \cdot v_1 + 0 \cdot v_2 \end{bmatrix} = 0 \quad v_2 = 2v_1$

$$\vec{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = 1$ $\quad \begin{bmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \cdot v_1 - v_2 \\ 0 \cdot v_1 - 2v_2 \end{bmatrix} = 0. \quad$ $v_2 = 0$
v_1 : arbitrary numb.

$$\vec{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

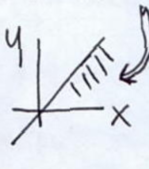
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}$$

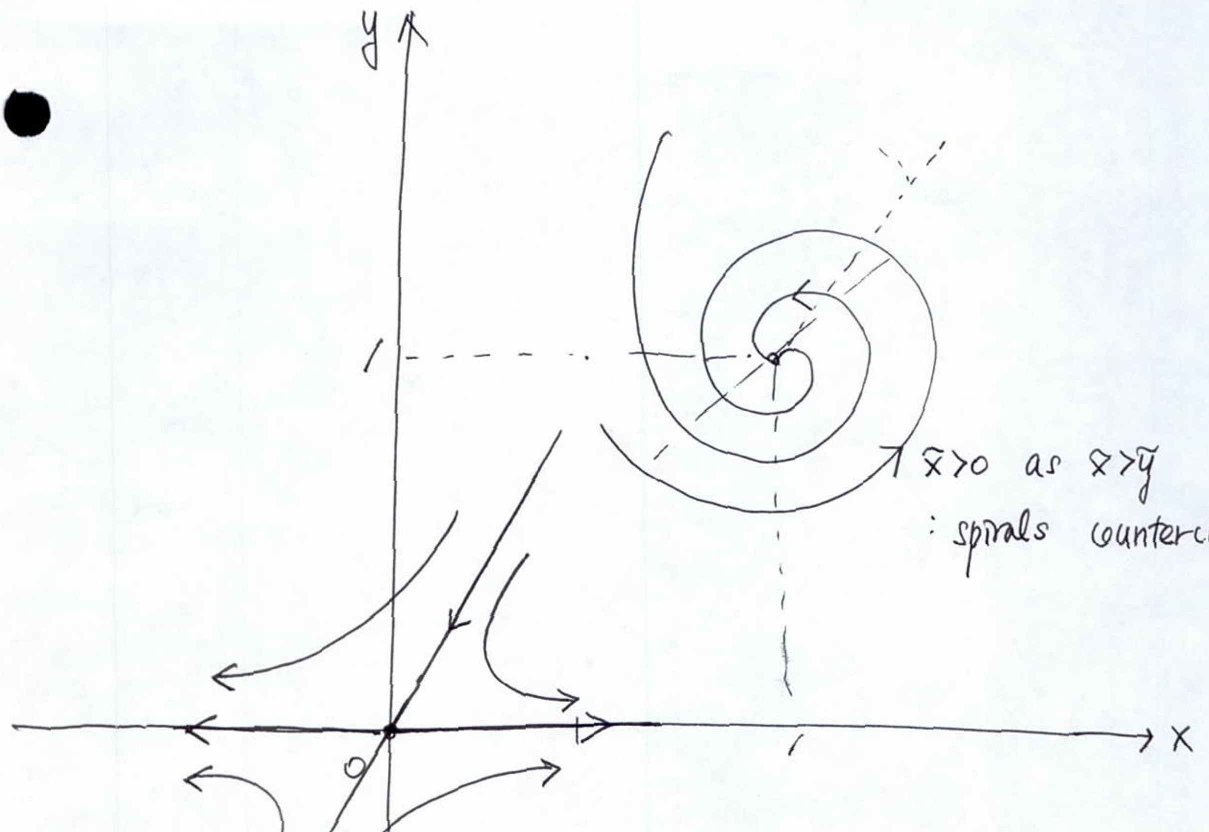
At $(1,1)$

We know that the trajectories spiral and get attracted to $(1,1)$.

spiral direction is determined by observing that

$\tilde{x}' = \tilde{x} - \tilde{y}$. $\quad \tilde{x}' > 0$ when $\tilde{x} > \tilde{y}$





$\tilde{x} > 0$ as $\tilde{x} > \tilde{y}$
: spirals counterclockwise

Engineering Mathematics I - Spring 2005

Examination 2

Total 150 points

June 14, 2005

1: 15 pts

Find a power series solution in powers of x of the following differential equation:

$$y'' - 4xy' + (4x^2 - 2)y = 0$$

2: 15 pts

Recalling that the standard form of Bessel's differential equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

find a general solution of the following equation in terms of Bessel functions. (Use the indicated substitutions.)

$$y'' + k^2x^4y = 0 \qquad (y = u\sqrt{x}, \quad \frac{1}{3}kx^3 = z)$$

3

(a: 10 pts) Find the Fourier sine series $\sum b_n \sin n\pi t$ for $1 - t$, on the interval (0,1].

(b: 10 pts) Find a Fourier since series solution $y(t)$ satisfying on (0,1) the ODE (k constant) $y'' + ky = 1 - t$, and the conditions $y(0) = 0$, $y(1) = 0$.

(c: 3 pts) Which term in the Fourier series solution in (b) would dominate if $k = 90$?

4: 15 pts

Solve the IVP $y'' - 2y' + y = 0$, $y(0) = 1$, $y'(0) = 0$ using the Laplace transform. No credit for other methods. You may use the table in the last page.

5

(a: 5 pts) Derive directly from the definitions the formula for the Laplace transform

$\mathcal{L}(f(t))$, where $f(t) = t$. (State explicitly any limit you use, but you need not prove it.)

(b: 5 pts) Let $F(s) = \mathcal{L}(f(t))$, and $a > 0$. Find a formula for $\mathcal{L}(f(at))$ in terms of $F(s)$.

6

(a: 10 pts) Solve by the Laplace transform: $y' = y - c\delta(t-1)$, $y(0) = 1$, where c is a constant and $\delta(t)$ is the Dirac delta function. Express your answer in the "cases" format,

$$y = \begin{cases} f(t), & 0 \le t < a \\ g(t), & t \ge a \end{cases}$$

(b: 2 pts) For what value of the constant c will the solution be 0 for large t?

7

Consider the function $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin(t-\pi), & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

(a: 5 pts) Use step functions to give an alternate description of this function.

(b: 5 pts) Use the result of part (a) or another method to compute the Laplace transform of $f(t)$.

8

(a: 15 pts) In Los Angeles, a bicycle race is held every year to climb the steepest hill in the city. On the hill, the elevation is written as $f(x,y) = 4 - \frac{2}{3}\sqrt{x^2+y^2}$, $0 \le z \le 4$. Mathematically derive the direction of the shortest path to reach the apex of the hill using the concept of gradient. (5 points for a mere physical argument without using gradient)

(b: 5 pts) Find a unit normal vector for the following surface at the given point:

$$x^2 + y^2 + 2z^2 = 26, \qquad P : (2, 2, 3)$$

9

Evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dA$.

(a: 15 pts) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, $\quad S : \mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + 3v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le 2\pi$.

(b: 15 pts) $\mathbf{F} = [x^3, y^3, z^3]$, $\quad S$: the sphere $x^2+y^2+z^2 = 9$. Use the divergence theorem.

Table of Laplace Transforms

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$
$e^{at} f(t)$	$F(s-a)$
f'	$sF(s) - f(0)$
$f(t-a)u(t-a)$	$e^{-as} F(s)$
$tf(t)$	$-F'(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

Engineering Mathematics I

Exam 2 Solutions 6/14/2005

Prob. 1 $y'' - 4xy' + (4x^2 - 2)y = 0$

$$\begin{cases} y = \sum_{m=0}^{\infty} a_m x^m \\ y' = \sum_{m=1}^{\infty} a_m \cdot m x^{m-1} \\ y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} \end{cases}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=1}^{\infty} 4m a_m x^m + \sum_{m=0}^{\infty} (4a_m x^{m+2} - 2a_m x^m) = 0.$$

x^0: $2a_2 + (-2a_0) = 0$: $a_2 = a_0$

x^1: $6a_3 - 4a_1 - 2a_1 = 0$: $a_3 = a_1$

x^2 or over : $\sum_{m=2}^{\infty} [(m+2)(m+1)a_{m+2} - 4m a_m + 4a_{m-2} - 2a_m] x^m = 0$

$$(m+2)(m+1)a_{m+2} = (4m+2)a_m - 4a_{m-2}$$

$$a_{m+2} = \frac{1}{(m+1)(m+2)} [(4m+2)a_m - 4a_{m-2}]$$

$m=2$: $a_4 = \frac{1}{3 \cdot 4}(10 \cdot a_2 - 4a_0) = \frac{1}{2} a_0$

$m=3$: $a_5 = \frac{1}{4 \cdot 5}(14 a_3 - 4a_1) = \frac{1}{2} a_1$

$m=4$: $a_6 = \frac{1}{5 \cdot 6}(18 a_4 - 4a_2) = \frac{1}{6} a_0 = \frac{1}{3!} a_0$

$m=5$: $a_7 = \frac{1}{6 \cdot 7}(22 a_5 - 4a_3) = \frac{1}{6} a_0 = \frac{1}{3!} a_0$

$m=6$: $a_8 = \frac{1}{7 \cdot 8}(26 a_6 - 4a_4) = \frac{1}{7 \cdot 8}(\frac{13}{3} a_0 - 2a_0)$

$$= \frac{1}{7 \cdot 8} \cdot \frac{7}{3} a_0 = \frac{1}{24} a_0 = \frac{1}{4!} a_0$$

$$\therefore\ y = a_0\left(1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots\right)$$

$$+ a_1\left(x + x^3 + \frac{1}{2!}x^5 + \frac{1}{3!}x^7 + \cdots\right)$$

$$= a_0 e^{x^2} + a_1 x e^{x^2}$$

$$= (a_0 + a_1 x)e^{x^2}$$

Prob. 2. $\quad y'' + k^2 x^4 y = 0.$

$$y = u\sqrt{x}, \quad \frac{1}{3}kx^3 = z$$

$$\frac{dy}{dx} = \sqrt{x}\frac{du}{dx} + \frac{1}{2}x^{-1/2}u \qquad \left(\frac{du}{dx} = u'\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}x^{-1/2}u' + \sqrt{x}u'' + \frac{1}{2}x^{-1/2}u' - \frac{1}{4}x^{-3/2}u$$

$$= \sqrt{x}u'' + x^{-1/2}u' - \frac{1}{4}x^{-3/2}u$$

Then we get

$$\sqrt{x}u'' + x^{-1/2}u' - \frac{1}{4}x^{-3/2}u + k^2x^{9/2}u = 0$$

multiplying by $x^{3/2}$,

$$x^2u'' + xu' + \left(k^2x^6 - \frac{1}{4}\right)u = 0$$

Now we substitute z for x

$$\frac{d}{dx} = \left(\frac{dz}{dx}\right)\frac{d}{dz} = kx^2\frac{d}{dz}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(kx^2\frac{d}{dz}\right) = 2kx\frac{d}{dz} + (kx^2)^2\frac{d^2}{dz^2}$$

Therefore,

$$2kx^3 \frac{du}{dz} + k^2x^6 \frac{d^2u}{dz^2} + kx^3 \frac{du}{dz} + (9z^2 - \frac{1}{4}) u = 0$$

$$9z^2 \frac{d^2u}{dz^2} + \underbrace{3kx^3}_{=9z} \frac{du}{dz} + (9z^2 - \frac{1}{4}) u = 0$$

$$z^2 \frac{d^2u}{dz^2} + z \frac{du}{dz} + (z^2 - \frac{1}{36}) u = 0$$

$$\therefore u = A J_{\frac{1}{6}}(z) + B Y_{\frac{1}{6}}(z) = \frac{1}{\sqrt{x}} y$$

$$\therefore y = \sqrt{x} \left[A J_{\frac{1}{6}}(\tfrac{1}{3}kx^3) + B Y_{\frac{1}{6}}(\tfrac{1}{3}kx^3) \right]$$

Prob. 3

(a) $f(t) = 1 - t = \sum_{n=1}^{\infty} b_n \sin m\pi t \qquad 0 < t \leq 1$

We use orthogonality to obtain b_n :

$$\int_0^1 (1-t) \sin m\pi t \, dt = \int_0^1 \sin m\pi t \sum_{n=1}^{\infty} b_n \sin m\pi t \, dt$$

$$= b_m \int_0^1 \sin^2 m\pi t \, dt$$

cf.) $\int_0^1 \sin^2 m\pi t \, dt = \frac{1}{2} \int_0^1 (1 - \cos 2m\pi t) dt$

$$= \frac{1}{2} \left[t - \frac{1}{2m\pi} \sin 2m\pi t \right]_0^1$$

$$= \frac{1}{2}.$$

$$\int_0^1 \overset{u}{t} \, \overset{v'}{\sin m\pi t} \, dt = -\left[\frac{1}{m\pi} t \cos m\pi t \right]_0^1 + \int_0^1 \frac{1}{m\pi} \cos m\pi t \, dt$$

$$= -\frac{1}{m\pi} \cos m\pi = \frac{1}{m\pi} (-1)(-1)^m = \frac{(-1)^{m+1}}{m\pi}$$

$$\int_0^1 \sin m\pi t = -\frac{1}{m\pi}\Big[\cos m\pi t\Big]_0^1 = \frac{1}{m\pi}\left(1-(-1)^m\right)$$

$$\to \frac{1}{m\pi}\Big[1-\underbrace{\{(-1)^m+(-1)^{m+1}\}}_{=0}\Big] = \frac{1}{2} b_m$$

$$b_m = \frac{2}{m\pi}$$

$$\therefore\ 1-t = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi t. \qquad 0<t\leq 1.$$

(b) $\quad y''+ky = 1-t.$

$$\text{Let } y=\sum_{m=1}^{\infty} a_n \sin m\pi t \quad \leftarrow y(0)=y(1)=0$$

$$y'=\sum_{m=1}^{\infty} a_m(m\pi)\cos m\pi t$$

$$y''=-\sum_{m=1}^{\infty}(m\pi)^2 a_m \sin m\pi t$$

from (a)

$$\to \sum_{m=1}^{\infty}\left(-m^2\pi^2+k\right)a_m\sin m\pi t = \sum_{m=1}^{\infty}\frac{2}{m\pi}\sin m\pi t$$

$$\therefore\ a_m = \frac{2}{m\pi}\cdot\frac{1}{(k-m^2\pi^2)}.$$

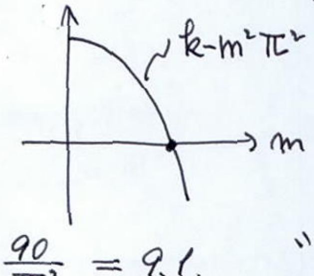
$$\therefore\ y(t)=\sum_{m=1}^{\infty}\frac{2}{m\pi(k-m^2\pi^2)}\sin m\pi t$$

(c) $k=90$ $\Rightarrow$ dominant term corresponds to "m" which minimizes $|k-m^2\pi^2|$

$m=1:\ \dfrac{1}{90-\pi^2}$

$m=2:\ \dfrac{1}{90-4\pi^2}$

$\vdots$



$$m^2 \approx \frac{90}{\pi^2} = 9.1. \qquad \text{"}m=3\text{"}$$

Prob. 4. $y''-2y'+y=0$

$$\mathcal{L}(y') = sY - \cancelto{1}{y(0)}$$

$$\mathcal{L}(y'') = s^2Y - s\cancelto{1}{y(0)} - \cancelto{0}{y'(0)}$$

$$s^2Y - s - 2(sY-1) + Y = 0$$

$$(s^2-2s+1)Y = s-2$$

$$Y = \frac{s-2}{(s-1)^2} = \frac{-1}{(s-1)^2} + \frac{1}{s-1}$$

$$y = \mathcal{L}^{-1}(Y) = -te^t + e^t = (1-t)e^t$$

Prob. 5.

(a) $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad s>0$

$$\mathcal{L}(t) = \int_0^\infty \underset{v'}{e^{-st}} \cdot \underset{u}{t}\, dt = -\frac{1}{s}\left[te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{1}{s^2}(0-1) = \frac{1}{s^2}$$

(b) $\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$

Let $\tau = at$. $\quad d\tau = a\,dt$

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-s\frac{\tau}{a}} f(\tau)\frac{1}{a}\,d\tau = \frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})\tau} f(\tau)\,d\tau$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

Prob. 6.

(a) $y' = y - c\,\delta(t-1)$, $y(0) = 1$

$sY - y(0) = Y - c\mathcal{L}\{\delta(t-1)\}$

$sY - 1 = Y - ce^{-s}$

$(s-1)Y = 1 - ce^{-s}$

$Y = \dfrac{1-ce^{-s}}{s-1} = \dfrac{1}{s-1} - \dfrac{c}{s-1}e^{-s}$

$y = e^t - ce^{t-1}u(t-1)$

$= \begin{cases} e^t . & 0 \le t < 1 \\ e^t - ce^{t-1} . & t \ge 1 \end{cases}$

(b) $e^t - ce^{t-1} = 0$

$ce^{t-1} = e^t$

$c = e^{t-t+1} = e$

Prob. 7.

(a) $f(t) = \sin(t-\pi)\,[u(t-\pi) - u(t-2\pi)]$

(b) $\mathcal{L}\{\sin(t-\pi)\,u(t-\pi)\} = e^{-\pi s} \cdot \dfrac{1}{s^2+1}$

$\sin(t-\pi) = \sin(t-2\pi+\pi) = -\sin(t-2\pi)$

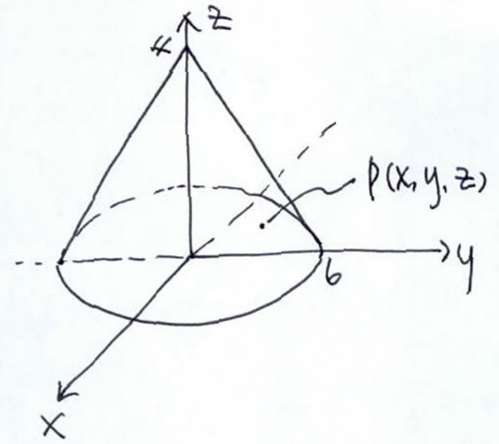
$\mathcal{L}\{\sin(t-\pi)\,u(t-2\pi)\} = \mathcal{L}\{-\sin(t-2\pi)\,u(t-2\pi)\}$

$= -e^{-2\pi s}\dfrac{1}{s^2+1}.$

$\mathcal{L}\{f(t)\} = (e^{-\pi s} + e^{-2\pi s})\,\dfrac{1}{s^2+1}$

Prob. 8

(a)



At a point P on the hill surface, the distance (l) to the top is written as

$$g = l^2 = x^2 + y^2 + (z-4)^2 .$$

The direction of the shortest path is the direction in which l^2 changes the most rapidly.

Therefore, the direction is grad $g = \nabla g$

$$\therefore \nabla g = 2x\,\hat{i} + 2y\,\hat{j} + 2(z-4)\,\hat{k}$$

$$= 2\left(x\hat{i} + y\hat{j} + (z-4)\hat{k}\right)$$

For the vector to indicate $(0,0,4)$ from (x,y,z) the direction $\bar{u}$ should be:

$$\bar{u} = -\left[x\hat{i} + y\hat{j} + (z-4)\hat{k}\right]$$

where $z = 4 - \frac{2}{\sqrt{3}}\sqrt{x^2+y^2}$

$0 \le x \le 6$

$0 \le y \le 6 .$

(b) Equation for surface: $f = x^2 + y^2 + 2z^2 - 26 = 0$

$\hat{n} = \frac{1}{|\nabla f|} \nabla f$ $\qquad P(2,2,3)$

$\nabla f = 2x\,\hat{i} + 2y\,\hat{j} + 4z\,\hat{k}$

$= 2(x\hat{i} + y\hat{j} + 2z\hat{k}) = 2(2\hat{i} + 2\hat{j} + 6\hat{k})$

$= 2(\hat{i} + \hat{j} + 3\hat{k})$

$|\nabla f| = 2\sqrt{1+1+9} = 2\sqrt{11}$

$\therefore\ \hat{n} = \frac{1}{\sqrt{11}}(\hat{i} + \hat{j} + 3\hat{k})$

Prob. 9. $\iint_S \bar{F} \cdot \hat{n}\, dA = \iint_R \bar{F} \cdot \bar{N}\, du\,dv$

$\bar{N} = \bar{r}_u \times \bar{r}_v$

(a) $S: \bar{r} = u\cos v\,\hat{i} + u\sin v\,\hat{j} + 3v\,\hat{k}$

$\bar{r}_u = \cos v\,\hat{i} + \sin v\,\hat{j}$

$\bar{r}_v = -u\sin v\,\hat{i} + u\cos v\,\hat{j} + 3\hat{k}$

$$\bar{N} = \bar{r}_u \times \bar{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 3 \end{vmatrix}$$

$= \hat{i}(3\sin v) - \hat{j}(3\cos v) + \hat{k}(\overbrace{u\cos^2 v + u\sin^2 v}^{=u})$

$\bar{F} = u^2\cos^2 v\,\hat{i} + u^2\sin^2 v\,\hat{j} + 9v^2\,\hat{k}$

$\bar{F} \cdot \bar{N} = 3u^2\sin v\cos^2 v - 3u^2\sin^2 v\cos v + 9uv^2$

$= 3u^2\sin v\cos v(\cos v - \sin v) + 9uv^2$

$$I = \iint_R \left[3u^2 \sin v \cos v (\cos v - \sin v) + 9uv^2\right] du\,dv$$

$$= \int_0^{2\pi} \int_0^1 \left[3u^2 \sin v \cos v (\cos v - \sin v) + 9uv^2\right] du\,dv$$

$$= \int_0^{2\pi} \left[\sin v \cos v (\cos v - \sin v) + \frac{9}{2} v^2\right] dv$$

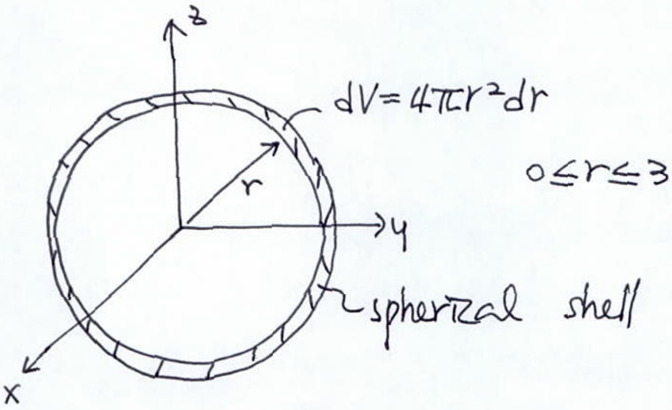
cf) $\int_0^{2\pi} \sin v \cos^2 v \, dv = \int_0^{2\pi} \sin^2 v \cos v \, dv = 0$

$$I = \frac{9}{6}\left[v^3\right]_0^{2\pi} = \frac{\cancel{9}^{3}}{\cancel{6}_{2}} \cdot 8\pi^3 = 12\pi^3$$

(b) Divergence theorem.

$$\iint_S \bar{F} \cdot \hat{n}\, dA = \iiint_V \nabla \cdot \bar{F}\, dV$$

$$\nabla \cdot \bar{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3r^2$$



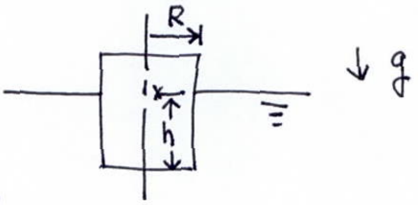
$$I = \int_0^3 3r^2 \cdot 4\pi r^2 dr = 12\pi \frac{1}{5}\left[r^5\right]_0^3 = \frac{12}{5}\pi (243)$$

$$= \frac{2916}{5}\pi \cong 1832$$

Quiz 2 (2006)

Prob. 2

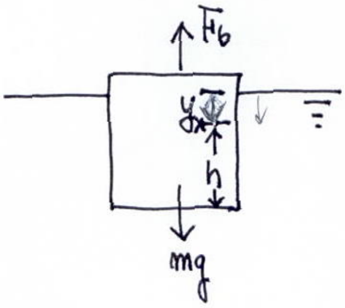
· at equilibrium



$$mg = \rho_w g \pi R^2 h_{eq}$$

$$h_{eq} = \frac{m}{\rho_w \pi R^2}$$

· disturbed



$$\downarrow \Sigma F_{ext} = mg - \rho_w g \pi R^2 (h+y) = m\ddot{y}$$

$$m\ddot{y} + \underbrace{\rho_w g \pi R^2}_{k} y = 0$$

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}$$

2006 final #3

(a) $x = \sum_{n=1}^{\infty} b_n \sin n\pi x.$

using orthogonality,

$$\int_0^1 x \sin m\pi x \, dx = \int_0^1 \underbrace{\sin m\pi x \sum_{n=1}^{\infty} b_n \sin n\pi x \, dx}_{} \quad 4$$

$$= b_m \int_0^1 \sin^2 m\pi x \, dx$$

$$\int_0^1 \sin^2 m\pi x \, dx = \frac{1}{2}$$

우변 $= \frac{b_m}{2}$ 2

$$\int_0^1 x \sin m\pi x \, dx = \frac{(-1)^{m+1}}{m\pi}$$: 2 (부분적분)

$$\frac{b_m}{2} = \frac{(-1)^{m+1}}{m\pi}$$

$$b_m = \frac{2}{m\pi}(-1)^{m+1}$$: 2 (답)

$$\therefore x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi x$$

(b) $y'' + ky = x$

$$y = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

$$y'' = -\sum_{n=1}^{\infty} a_n (n\pi)^2 \sin n\pi x$$

좌변 : 5 A=B

$$\sum_{n=1}^{\infty} (-n^2\pi^2 + k) a_n \sin n\pi x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi x$$: 2

$$a_n = (-1)^{n+1} \frac{2}{n\pi(k - n^2\pi^2)}$$: 3 (a_n)

$$\therefore y(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi(k-n^2\pi^2)} \sin n\pi x$$

(c) $n=3$. : 5.

Midterm solutions. 2006 spring

3 (i) $y'' + 9y' + 14y = 0$

$y = e^{\lambda t}$

$\lambda^2 + 9\lambda + 14 = (\lambda+2)(\lambda+7) = 0.$ $\lambda = -2, -7$

$y = c_1 e^{-2t} + c_2 e^{-7t}.$ $\rightarrow 0$ as $t \rightarrow \infty.$ (A)

3 2

(ii) $y'' + 5y' + 6y = 6 + e^{-2t}$

$y = y_h + y_p$ (1)

$y_h = e^{\lambda t}.$ $\lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3) = 0$

$\lambda = -2, -3$

$y_h = c_1 e^{-2t} + c_2 e^{-3t}$ (1)

$y_p = A + Bte^{-2t}$

$y_p' = Be^{-2t} - 2Bt e^{-2t}$

$y_p'' = -2Be^{-2t} - 2Be^{-2t} + 4Bte^{-2t} = -4Be^{-2t} + 4Bte^{-2t}$

$e^{-2t}(-4B + 5B) + te^{-2t}(4B - 10B + 6B) + 6A = 6 + e^{-2t}$

$A = 1.$ $B = 1$. $y_p = 1 + te^{-2t}$ (2)

Gen. sol. $y = c_1 e^{-2t} + c_2 e^{-3t} + 1 + te^{-2t}.$ (~~B~~) (1)

C

(iii) $y'' - 2y' + 5y = e^{-t}$

$y = y_h + y_p$

$y_h = e^{\lambda t}.$ $\lambda^2 - 2\lambda + 5 = 0$ $\lambda = 1 \pm \sqrt{1-5} = 1 \pm 2i$

$y_h = e^{t}(c_1 \cos 2t + c_2 \sin 2t)$ (2)

$y_p = Ae^{-t}.$ $y_p' = -Ae^{-t}.$ $y_p'' = Ae^{-t}$

$A + 2A + 5A = 1.$ $8A = 1.$ $A = \frac{1}{8}.$

$\therefore y = e^{t}(c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{8}e^{-t}.$ (2) (B) (1)

4. $y_{p_1} = \frac{x}{2} + x^2$. $y_{p_1}' = \frac{1}{2} + 2x$, $y_{p_1}'' = 2$

$y_{p_2} = \frac{x}{2} - 3x^2$. $y_{p_2}' = \frac{1}{2} - 6x$. $y_{p_2}'' = -6$

each of them satisfies $x^2y'' + p(x)y' + 6y = x$

(i) $y_{p_1} \Rightarrow 2x^2 + (\frac{1}{2} + 2x)p(x) + 3x + 6x^2 = x$

$$(\tfrac{1}{2} + 2x)\,p(x) = -8x^2 - 2x \quad \cdots (1)$$

$y_{p_2} \Rightarrow -6x^2 + (\frac{1}{2} - 6x)p(x) + 3x - 18x^2 = x$

$$(\tfrac{1}{2} - 6x)\,p(x) = 24x^2 - 2x \quad \cdots (2)$$

代入 3

$(1) - (2) \Rightarrow \quad 8x \cdot p(x) = -32x^2$.

$$p(x) = -4x$$

2

(ii) $x^2y'' - 4xy' + 6y = x$

$y_h = x^m$

$m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3) = 0$

$m = 2, 3$

$y_h = c_1x^2 + c_2x^3$ (3)

$y_p = \frac{x}{2}$. from the problem statement (2)

Gen. sol. $\Rightarrow \quad y = c_1x^2 + c_2x^3 + \frac{x}{2}$