Engineering Mathematics 1 (010.140)

1st Midterm Examination

#1. Find $y = f(p)$ by solving $p^2 + py = x, p = y'$

Sol) $y = \frac{1}{p}x - p$ (*)

x 에 관해 미분; $y' = \frac{1}{p} - \frac{1}{p^2}x\frac{dp}{dx} - \frac{dp}{dx} = p$

$p^3 - p + (x + p^2)\frac{dp}{dx} = 0$

$\therefore\ \frac{dx}{dp} + \frac{x}{p^3-p} = -\frac{p}{p^2-1}$ (**)

Integrating factor F(p); $F(p) = e^{\int \frac{1}{p^3-p}dp} = \frac{\sqrt{p^2-1}}{p}$

(**)의 양변에 integrating factor 를 곱하면, 좌변; $\frac{\sqrt{p^2-1}}{p}\left(\frac{dx}{dp} + \frac{x}{p^3-p}\right) = \frac{d}{dp}\left(\frac{\sqrt{p^2-1}}{p}x\right)$

우변; $\frac{\sqrt{p^2-1}}{p}\left(-\frac{p}{p^2-1}\right) = -\frac{1}{\sqrt{p^2-1}}$

$\therefore \frac{d}{dp}\left(\frac{\sqrt{p^2-1}}{p}x\right) = -\frac{1}{\sqrt{p^2-1}}$

양변을 적분하면, $\frac{\sqrt{p^2-1}}{p}x = -\ln\left[p + \sqrt{p^2-1}\right] + C$

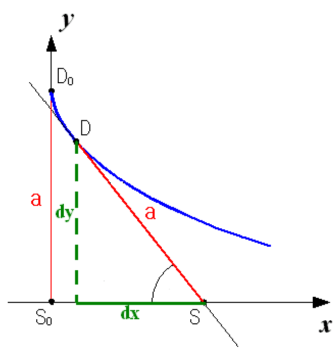
$\therefore x = -\frac{p}{\sqrt{p^2-1}}\ln\left[p + \sqrt{p^2-1}\right] + \frac{p}{\sqrt{p^2-1}}C$ (***)

(***)을 (*)에 대입하여 y 를 p 에 대해 정리하면,

$\therefore\ y = -\frac{1}{\sqrt{p^2-1}}\ln\left[p + \sqrt{p^2-1}\right] + \frac{1}{\sqrt{p^2-1}}C - p$

#2. The destroyer D (the pursuer) pursues the ship S (the target), that is, moves in the direction of S at all times. Assume that S moves along the x-axis and the distance a from D to S is constant. Show that $y' = -\frac{y}{\sqrt{a^2-y^2}}$. Sketch a direction field (for a=1 [nautical mile]) and the solution satisfying y(0)=1 (This curve is called a tractrix, from Latin trahere, meaning "to pull"). Separating variables, show that $x = -\int y^{-1}\sqrt{a^2-y^2}\,dy = -\sqrt{a^2-y^2} + a\ln\left|y^{-1}(a+\sqrt{a^2-y^2}\right| + C$

Sol)

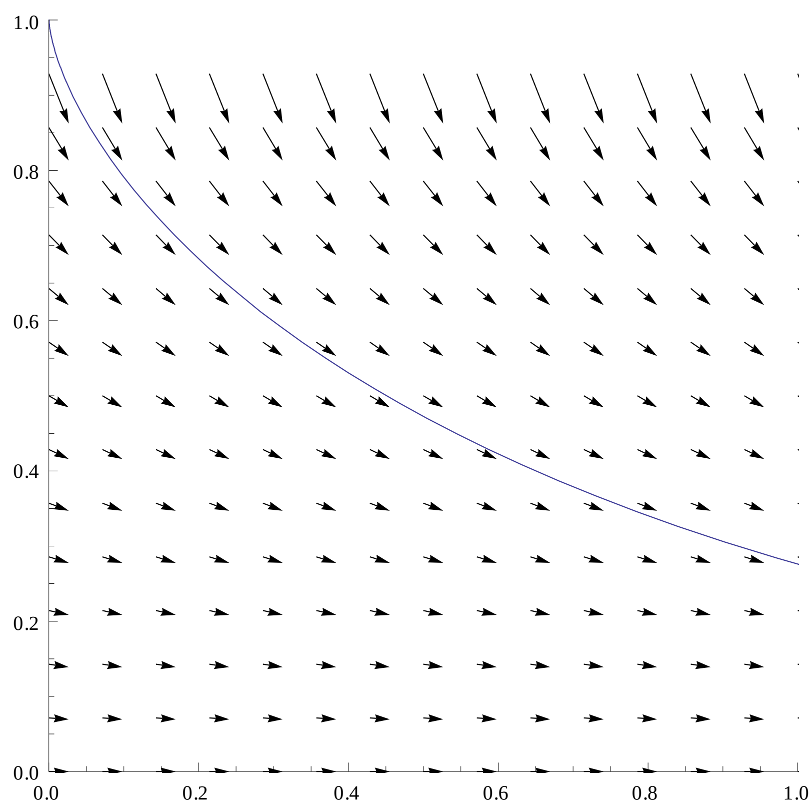


D 와 S 의 거리는 항상 a 이므로,

$\Delta y = -y$라 두면, $\Delta x = \sqrt{a^2-y^2}$

$$\therefore\ y' = \frac{dy}{dx} = \frac{\Delta y}{\Delta x} = -\frac{y}{\sqrt{a^2-y^2}} \qquad (*)$$

a=1 일 때, direction field (실선은 a=1, y(0)=1 일 때의 해)



(*)을 변수분리법의 꼴로 고치면, $dx = -\frac{\sqrt{a^2-y^2}}{y}dy$

$$\int dx = -\int \frac{\sqrt{a^2-y^2}}{y}dy$$

$$\therefore x = -\int \frac{\sqrt{a^2-y^2}}{y}dy$$

부분적분($\int uv' = uv - \int u'v$)을 이용하여,

$$u = \frac{1}{y},\ v' = \sqrt{a^2-y^2};\qquad u' = -\frac{1}{y^2},\ v = \frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)$$

$$\int \frac{\sqrt{a^2-y^2}}{y}dy = \frac{1}{y}\left(\frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right) - \int\left(-\frac{1}{y^2}\right)\left(\frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right)dy$$

$$= \frac{1}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}y^{-1}\sin^{-1}\left(\frac{y}{a}\right) + \int\left(\frac{1}{2}\frac{\sqrt{a^2-y^2}}{y} + \frac{a^2}{2}y^{-2}\sin^{-1}\left(\frac{y}{a}\right)\right)dy$$

$$= \frac{1}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}y^{-1}\sin^{-1}\left(\frac{y}{a}\right) + \int\frac{1}{2}\frac{\sqrt{a^2-y^2}}{y}dy + \int\frac{a^2}{2}y^{-2}\sin^{-1}\left(\frac{y}{a}\right)dy$$

$$\therefore\ \frac{1}{2}\int\frac{\sqrt{a^2-y^2}}{y}dy = \frac{1}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}y^{-1}\sin^{-1}\left(\frac{y}{a}\right) + \int\frac{a^2}{2}y^{-2}\sin^{-1}\left(\frac{y}{a}\right)dy$$

$$\int\frac{\sqrt{a^2-y^2}}{y}dy = \sqrt{a^2-y^2} + a^2y^{-1}\sin^{-1}\left(\frac{y}{a}\right) + \int a^2y^{-2}\sin^{-1}\left(\frac{y}{a}\right)dy$$

$$= \sqrt{a^2-y^2} + a^2y^{-1}\sin^{-1}\left(\frac{y}{a}\right) + \left(-a^2y^{-1}\sin^{-1}\left(\frac{y}{a}\right) - \int\left(-\frac{a^2}{y}\right)\frac{a}{\sqrt{a^2-y^2}}dy\right)$$

$$= \sqrt{a^2-y^2} + \int\frac{a^2}{y}\frac{a}{\sqrt{a^2-y^2}}dy$$ ($y = a\sin\theta, dy = a\cos\theta d\theta$로 치환)

$$= \sqrt{a^2-y^2} + a\int\frac{1}{\sin\theta}d\theta$$

$$= \sqrt{a^2-y^2} + a\ln\left|\tan\frac{\theta}{2}\right| + C$$

($\sin\frac{\theta}{2} = \frac{y}{2a\cos\frac{\theta}{2}}$, $\cos^2\frac{\theta}{2} = \frac{\cos\theta+1}{2}$, $\cos\theta = \sqrt{1-\frac{y^2}{a^2}}$를 이용하여 치환)

$$= \sqrt{a^2-y^2} + a\ln\left|\frac{y}{\sqrt{a^2-y^2}+a}\right| + C$$

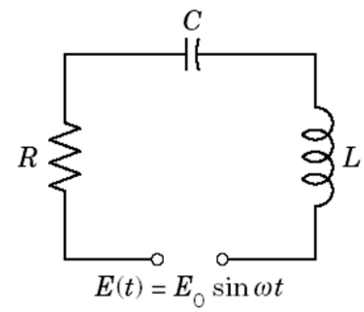
$$\therefore x = -\int\frac{\sqrt{a^2-y^2}}{y}dy = -\sqrt{a^2-y^2} + a\ln\left|\frac{\sqrt{a^2-y^2}+a}{y}\right| + C$$

#3. Advanced engineering mathematics, KREYSZIG, 9th, 2.9-9

(1) Find the governing equation of the electric circuit with respect to the charge, Q(t).

Sol) $LI'' + RI' + \frac{1}{C}I = E' = Q(t)$

(2) Find the general solution of the governing equation of (1) by considering the conditions for the circuit to be overdamped, critically damped and underdamped.



Sol) overdamped condition; $R > 2\sqrt{\frac{L}{C}}$

critically damped condition; $R = 2\sqrt{\frac{L}{C}}$

underdamped condition; $R < 2\sqrt{\frac{L}{C}}$

(3) What is the critical resistance Rcrit?

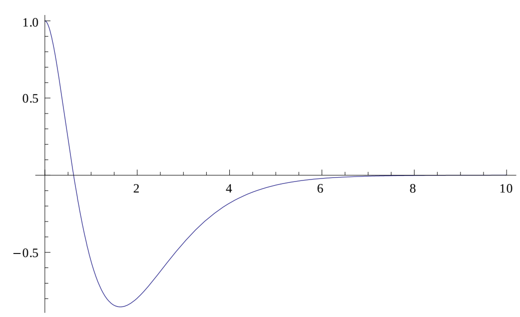
Sol) $R_{crit} = 2\sqrt{\frac{L}{C}}$

#4. Advanced engineering mathematics, KREYSZIG, 9th, 2.7-ex2 / 2.10-18(a) / 3.3-11

(1) Solve the initial value problem and graph the solution

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, y(0) = 1, y'(0) = 0.$$

Sol) $y = (1 + 1.5x - 5x^2)e^{-1.5x}$



(2) Solve $y'' + sy' - 15y = 17\sin 5x$ by undetermined-coefficient method and by variation of parameters.

Sol) $y = C_1e^{3x} + C_2e^{-5x} - 0.1\cos 5x - 0.4\sin 5x$

(3) Solve the initial value problem

$$(x^3D^3 - x^2D^2 - 7xD + 16I)y = 9x\ln x, y(1) = 6, Dy(1) = 18, D^2y(1) = 65$$

Sol) $y = -x^2 + x^{-2} + 5x^4 + x\ln x + x$