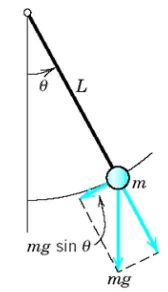
Engineering Mathematics 1 (010.140)

2nd Midterm Examination

#1. Advanced engineering mathematics, KREYSZIG, 9th, 4.5-Ex.1

A pendulum consisting of a body of mass m (the bob) and a rod of length L. Assume that the mass of the rod and air resistance are negligible.



(1) Set up the mathematical model.

Sol) $\theta'' + k\sin\theta = 0$

(2) Determine the locations and types of the critical point.

Sol) ① Critical points (0, 0), ±(2π, 0), ±(4π, 0), ... , Linearization.

→(0,0) is a center, which is always stable.

Since sinθ is periodic with period 2π, the critical points (nπ, 0), n=±2, ±4, ... , are all centers.

② Critical points ±(π, 0), ±(3π, 0), ±(5π, 0), ... , Linearization.

→ (π,0) is a saddle point, which is always unstable.

Because of periodicity, the critical points (nπ, 0), n=±1, ±3, ... , are all saddle points.

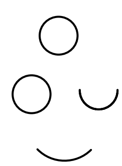
(3) Describe the physical meaning the phase plane below.

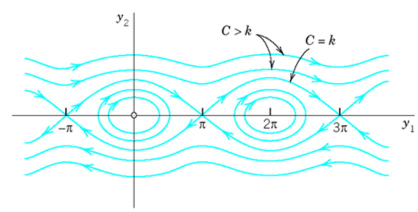
Sol) No damping.

i) $C > k$; circular motion

ii) $C = k$; circular or semicircular motion

iii) $C < k$; reciprocating motion





#2. Advanced engineering mathematics, KREYSZIG, 9th, 5.3-15

The associated Legendre functions $P_n^k(x)$ play a role in quantum physics. They are defined by $P_n^k(x) = (1-x^2)^{k/2}\dfrac{d^kP_n}{dx^k}$ and are solution of ODE $(1-x^2)y'' - 2xy' + \left[n(n+1) - \dfrac{k^2}{1-x^2}\right]y = 0$. Find $P_1^1(x)$ and $P_2^2(x)$.

Sol) $P_1(x) = x \Rightarrow P_1^1(x) = (1-x^2)^{1/2}\dfrac{dP_1}{dx} = \sqrt{1-x^2}$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow P_2^1(x) = (1-x^2)^{1/2}\frac{dP_2}{dx} = 3x\sqrt{1-x^2}$$

$$P_2^2(x) = (1-x^2)^{1/2}\frac{d^2P_2}{dx^2} = 3(1-x^2)$$

#3. Find a solution of $x^2y'' + (x^2 + \frac{5}{36})y = 0$.

Sol) Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+r}$,

$$x^2y'' + (x^2 + \frac{5}{36})y = \sum_{m=0}^{\infty}(m+r)(m+r-1)c_m x^{m+r} + \sum_{m=0}^{\infty} c_m x^{m+r+2} + \frac{5}{36}\sum_{m=0}^{\infty} c_m x^{m+r} = 0$$

(the coefficient of x^r)=0 ; $\left(r(r-1) + \frac{5}{36}\right)c_0 = 0$

$$\therefore r_1 = \frac{5}{6}, r_2 = \frac{1}{6}$$

(the coefficient of x^{r+s})=0; $\left((r+1)r + \frac{5}{36}\right)c_1 = 0$ if s=1 (*)

$(s+r)(s+r-1)c_s + c_{s-2} + \frac{5}{36}c_s = 0$ if s=2, 3, … (**)

i) $r = r_1 = \frac{5}{6}$,

in (*), $c_1 = 0$

in (**), $s\left(s + \frac{2}{3}\right)c_s + c_{s-2} = 0$

$\therefore c_3 = c_5 = c_7 = \ldots = 0$ $(\because c_1 = 0)$

let $s = 2p$, $c_{2p} = -\frac{3}{4}\frac{c_{2p-2}}{p(3p+1)}$ $(p = 1, 2, \ldots)$

$$c_2 = -\frac{3}{4}\frac{c_0}{4},\ c_4 = -\frac{3}{4}\frac{c_2}{2\times 7} = \left(\frac{3}{4}\right)^2 \frac{c_0}{2!4\times 7},$$

$$c_6 = -\frac{3}{4}\frac{c_4}{3\times 10} = -\left(\frac{3}{4}\right)^2 \frac{c_0}{3!4\times 7\times 10}$$

$$\therefore y_1(x) = c_0 \sum_{p=0}^{\infty} (-1)^p \left(\frac{3}{4}\right)^p \frac{x^{2p+5/6}}{p!1 \times 4 \times \ldots \times (3p+1)} = c_0 x^{\frac{5}{6}} \left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 - \ldots\right)$$

ii) $r = r_2 = \dfrac{1}{6}$,

in (*), $\quad c_1^* = 0$

in (**), $\quad s\left(s + \dfrac{2}{3}\right)c_s^* + c_{s-2}^* = 0$

$$\therefore c_3^* = c_5^* = c_7^* = \ldots = 0 \qquad (\because c_1^* = 0)$$

let $s = 2p$, $\qquad c_{2p}^* = -\dfrac{3}{4}\dfrac{c_{2p-2}^*}{p(3p-1)} \qquad (p = 1, 2, \ldots)$

$$\therefore y_2(x) = c_0^* + c_0^* \sum_{p=1}^{\infty} (-1)^p \left(\frac{3}{4}\right)^p \frac{x^{2p+1/6}}{p!2 \times 5 \times \ldots \times (3p-1)} = c_0^* x^{\frac{1}{6}} \left(1 - \frac{3}{8}x^2 + \ldots\right)$$

$$\therefore y(x) = y_1(x) + y_2(x) = c_0 x^{\frac{5}{6}} \left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 - \ldots\right) + c_0^* x^{\frac{1}{6}} \left(1 - \frac{3}{8}x^2 + \ldots\right)$$

#4. Advanced engineering mathematics, KREYSZIG, 9th, 5.5 Bessel's Equation. Bessel Function $J_\nu(x)$

Derive the solution of the first kind of order v of $x^2y'' + xy' + x^2y = \upsilon^2 y$.

Sol) $\quad J_n(x) = x^n \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$

#5. Advanced engineering mathematics, KREYSZIG, 9th, 6.5-Ex.4

Find y at a certain time t when the driving force is acting as $A\sin(\sqrt{k/m})t$ in an undamped mass-spring system, where the spring constant is k and the mass of the body attached to the spring is m.

Sol) $\quad y'' + \omega_0^2 y = A\sin\omega_0 t \quad$ where $\quad \omega_0^2 = k/m$.

$$y(t) = \frac{A\omega_0}{2\omega_0^2}\left(-t\cos\omega_0 t + \frac{\sin\omega_0 t}{\omega_0}\right) = \frac{A}{2\omega_0^2}(-\omega_0 t\cos\omega_0 t + \sin\omega_0 t)$$