Plasma Electrodynamics 1 April 9, 2009

Selected Solutions

(15 points)

1. If each of the following statements is true, please mark O in the parenthesis. If false or incorrect, mark X and make corrections of the false statement or word in the parenthesis.

1) Two vectors are *perpendicular* to each other if their cross product is zero, and vice versa. (X : cross --> dot or perpendicular --> parallel)

2) The basic rules of vector dot and cross products are independent of coordinate system. (O)

3) In the deductive approach, electromagnetic theorems and laws are developed from the fundamental postulates for an idealized model, and then verified with experimental observations. (O)

4) Maxwell's equations describe the electromagnetic field quantities generated by distributed space sources in a circuit model. (X : circuit --> field)

5) Coulomb C, which is the unit for charge, is a derived unit expressible in terms of MKSA units as A·s, and its dimension is *TI* . (O)

6) The gradient of a scalar field is a scalar point function. (X : scalar point --> vector point)

7) If $i=j$, then $\delta_{ij}=1$ and $\epsilon_{ijk}=0$. (O)

8) The divergence theorem transforms the *surface* integral of the divergence of a vector field to a closed *line* integral of the vector field, and vice versa.
(X : divergence theorem --> Stokes's theorem, divergence --> curl
or surface --> volume, line --> surface)

9) Helmholtz's theorem states that a vector field is determined if either its divergence or its curl is specified everywhere.
(X : either ..... or ..... is --> both ..... and ..... are)

10) If a vector field is curl-free, then it can be expressed as the gradient of a scalar field. (O)

(20 points)

2. Answer the following questions briefly.

1) What are the source quantities and four fundamental field quantities in the electromagnetic model? What are their SI units?

   Sources: charges q (C) and currents I (A)

   Fields: electric field intensity $\boldsymbol{E}$ (V/m), electric flux density $\boldsymbol{D}$ (C/m^2)

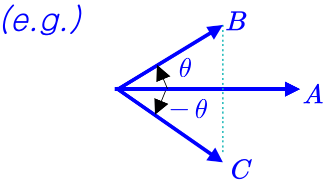
   magnetic field intensity $\boldsymbol{H}$ (A/m), magnetic flux density $\boldsymbol{B}$ (T)

2) Does $\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{A}\cdot\boldsymbol{C}$ imply $\boldsymbol{B}=\boldsymbol{C}$? Explain and show an example.

   Does $\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}$ imply $\boldsymbol{B}=\boldsymbol{C}$? Explain and show an example.

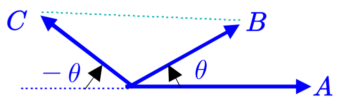
   In both cases, not necessarily $\boldsymbol{B}=\boldsymbol{C}$.

   (e.g.)



   $\boldsymbol{A}\cdot\boldsymbol{B}=AB\cos\theta=AC\cos\theta=AC\cos(-\theta)=\boldsymbol{A}\cdot\boldsymbol{C}$

   But, $\boldsymbol{B}\neq\boldsymbol{C}$



   $\boldsymbol{A}\times\boldsymbol{B}=AB\sin\theta=AC\sin\theta=AC\sin(\pi-\theta)=\boldsymbol{A}\times\boldsymbol{C}$

   But, $\boldsymbol{B}\neq\boldsymbol{C}$

3) Which of the following four scalar triple products has a different magnitude from the others? Explain.

   $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$, $\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B})$, $(\boldsymbol{A}\times\boldsymbol{C})\cdot\boldsymbol{B}$, $\boldsymbol{B}\cdot(\boldsymbol{a}_A\times\boldsymbol{A})$

   $\boldsymbol{B}\cdot(\boldsymbol{a}_A\times\boldsymbol{A})=0$ while $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=-\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B})=-(\boldsymbol{A}\times\boldsymbol{C})\cdot\boldsymbol{B}$

   By using the representations of summation convention and Levi-Civita symbol, prove that the scalar triple product $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$ can be calculated by the following determinant in Cartesian coordinate system.

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

   (Proof)

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=(\boldsymbol{A}\times\boldsymbol{B})_kC_k=\epsilon_{kij}A_iB_jC_k=\underset{i,j,k=x,y,z}{\epsilon_{ijk}A_iB_jC_k}=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

4) What are the metric coefficients in cylindrical coordinates?

Write $\boldsymbol{dl}$, $\boldsymbol{ds}_z$, dv, and ∇ in cylindrical coordinates.

Metric coefficients are conversion factors of differential coordinate variable changes to differential length changes.

In cylindrical coordinates,

Vector differential length $\boldsymbol{dl}$:

$$\boldsymbol{dl} = \hat{\boldsymbol{r}}\, dl_r + \hat{\boldsymbol{\phi}}\, \underline{dl_\phi} + \hat{\boldsymbol{z}}\, dl_z = \hat{\boldsymbol{r}}\, h_1 dr + \hat{\boldsymbol{\phi}}\, \underline{h_2 d\phi} + \hat{\boldsymbol{z}}\, h_3 dz$$

$$= \hat{\boldsymbol{r}}\, dr + \hat{\boldsymbol{\phi}}\, \underline{r d\phi} + \hat{\boldsymbol{z}}\, dz$$

Then, the metric coefficients in cylindrical coord. are

$$h_1 = 1,\ h_2 = r,\ h_3 = 1$$

Vector differential surface area $\boldsymbol{ds}_z$:

$$\boldsymbol{ds}_z = dl_r dl_\phi = \hat{\boldsymbol{z}}\, r dr\, d\phi \qquad (r-\phi \text{ plane})$$

Differential volume dv:

$$dv = dl_r dl_\phi dl_z = r dr\, d\phi\, dz$$

5) What is the physical definition of the curl of a vector field $\boldsymbol{A}$? Show that $\boldsymbol{A}$ is an irrotational field if $\boldsymbol{A} = \boldsymbol{a_R} f(R) \equiv \hat{\boldsymbol{R}} f(R)$ in spherical coordinates, where $f(R)$ is any function of the radial distance R.

$\nabla \times \boldsymbol{A}$ is a vector having the maximum circulation of $\boldsymbol{A}$ per unit area.

$$\nabla \times \boldsymbol{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{R}} & \hat{\boldsymbol{\theta}} R & \hat{\boldsymbol{\phi}} R\sin\theta \\ \partial/\partial R & \partial/\partial\theta & \partial/\partial\phi \\ f(R) & 0 & 0 \end{vmatrix}$$

$$= \hat{R}0 + \hat{\theta}\, R \underbrace{\cancel{\frac{\partial f(R)}{\partial\phi}}}_{0} + \hat{\phi}\, R\sin\theta \underbrace{\cancel{\frac{\partial f(R)}{\partial\theta}}}_{0} = 0 \quad \Rightarrow \quad \text{irrotational}$$

(12 points)

3. Assuming a vector field expressed in cylindrical coordinates to be $\boldsymbol{A} = \hat{r}\,(3\cos\phi) - \hat{\phi}\, 2r + \hat{z}\, z$.

1) What is the field at the point $P(4, 60^o, 1)$?

At $P(4, 60^o, 1)$, $\quad \boldsymbol{A}_p = \hat{r}\,(3\cos 60^o) - \hat{\phi}(2\times 4) + \hat{z}\,1 \;= \underline{\hat{r}\frac{3}{2} - \hat{\phi}\,8 + \hat{z}}$

2) Expressed the field $\boldsymbol{A_P}$ at P in Cartesian coordinates.

Using the transformation of vector from cylindrical to Cartesian coordinates,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos 60^o & -\sin 60^o & 0 \\ \sin 60^o & \cos 60^o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ -8 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ -8 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.68 \\ -2.70 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \underline{\boldsymbol{A}_p = \hat{x}\,7.68 - \hat{y}\,2.70 + \hat{z}\,1}$$

3) The cylindrical coordinates of another point Q are: $Q(3,\ 180^o,\ -1)$.

Determine the distance between the two points P and Q.

In Cartesian coordinates,

$$P(4\cos 60^o,\ 4\sin 60^o,\ 1) = P(2,\ 2\sqrt{3},\ 1)\ ,$$

$$Q(3\cos 180^o,\ 3\sin 180^o,\ -1) = Q(-3,\ 0,\ -1)$$

$$\therefore \quad \overline{PQ} = \sqrt{(-3-2)^2 + (0-2\sqrt{3})^2 + (-1-1)^2} = \sqrt{25+12+4} = \underline{\sqrt{41}}$$

(12 points)

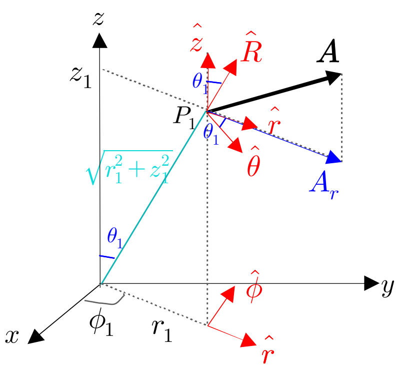
4. 1) Express the r-component, A_r , of a vector $\boldsymbol{A}$ at the point $P_1(r_1,\ \phi_1,\ z_1)$ in terms of A_R and A_θ in spherical coordinates.

In spherical coordinates, $\boldsymbol{A} = \hat{r} A_R + \hat{\theta} A_\theta + \hat{\phi} A_\phi$

$$A_r = \hat{r}\cdot\boldsymbol{A} = \underbrace{(\hat{r}\cdot\hat{R})}_{\cos(90^o-\theta_1) = \sin\theta_1} A_R + \underbrace{(\hat{r}\cdot\hat{\theta})}_{\cos\theta_1} + \underbrace{(\hat{r}\cdot\hat{\phi})}_{0} A_\phi$$

$$= A_R \sin\theta_1 + A_\theta \cos\theta_1$$

$$= \underline{A_R \frac{r_1}{\sqrt{r_1^2+z_1^2}} + A_\theta \frac{z_1}{\sqrt{r_1^2+z_1^2}}}$$



2) Denote the position vector to a point $P_2(x,\ y,\ z)$ by $\boldsymbol{R}$. Determine $\nabla(1/R)$ in spherical coordinates.

$\boldsymbol{R} = \hat{R} R$ where $R = |\boldsymbol{R}| = \sqrt{x^2+y^2+z^2}$

$$\therefore \quad \nabla\left(\frac{1}{R}\right) = \hat{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\right) = -\hat{R}\left(\frac{1}{R^2}\right) = \underline{-\frac{\hat{R}}{R^3}}$$

3) Find the divergence of the radial field, $f(\boldsymbol{R}) = \hat{\boldsymbol{R}}\, k/R^2$ where k is a constant.

Let $f(\boldsymbol{R}) = \hat{\boldsymbol{R}}\, k/R^2 \equiv \boldsymbol{B}$,

then $\nabla \cdot \boldsymbol{B} = \dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2 B_R\right) = \dfrac{1}{r^2}\dfrac{\partial}{\partial R}\left(R^2 \dfrac{k}{R^2}\right) = \underline{0}$

(5 points)

5\. Show that $\dfrac{\partial \hat{\boldsymbol{r}}}{\partial \phi} = \hat{\boldsymbol{\phi}}$ and $\dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\boldsymbol{r}}$ in cylindrical coordinates.

(These relations state that the differentiation of a base vector may lead to a new vector in a different direction in a curvilinear coordinate system)

Transformation of unit vectors from Cartesian to cylindrical coordinates:

$$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{\phi} = \hat{x}(-\sin\phi) + \hat{y}\cos\phi$$
$$\hat{z} = \hat{z}$$

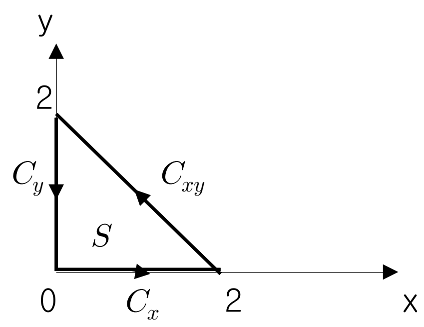
$$\therefore \quad \frac{\partial \hat{r}}{\partial \phi} = \hat{x}(-\sin\phi) + \hat{y}\cos\phi = \hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = \hat{x}(-\cos\phi) - \hat{y}\sin\phi = -\hat{r}$$

(16 points)

6\. Assume a vector field $\boldsymbol{A} = \hat{\boldsymbol{x}}(2x+y) - \hat{\boldsymbol{y}}\, y$.

1) Find $\oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$ around the triangular contour shown in the figure below.



$$d\boldsymbol{l} = \hat{x}\,dx + \hat{y}\,dy$$

$$\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = \oint (2x+y)\,dx - \oint y\,dy$$

Along C_x: $y=0,\ dy=0$

$$\Rightarrow \int_{C_x} \boldsymbol{A} \cdot d\boldsymbol{l} = \int_0^2 2x\,dx = x^2\Big|_0^2 = \underline{4}$$

Along C_{xy}: $y = 2-x,\ dy = -dx$

$$\Rightarrow \int_{C_{xy}} \boldsymbol{A} \cdot d\boldsymbol{l} = \int_2^0 (2x+2-x)\,dx + \int_2^0 (2-x)\,dx = \int_2^0 4\,dx = 4x\Big|_2^0 = \underline{-8}$$

Along C_x: $x=0,\ dx=0$

$$\Rightarrow \int_{C_y} \boldsymbol{A}\cdot d\boldsymbol{l} = -\int_2^0 y\,dy = -\left.\frac{y^2}{2}\right|_2^0 = 2$$

$$\therefore \oint_C \boldsymbol{A}\cdot d\boldsymbol{l} = 4-8+2 = \underline{-2}$$

2) Find $\int_S (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{s}$ over the triangular area.

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+y & -y & 0 \end{vmatrix} = -\hat{z}, \qquad d\boldsymbol{s} = \hat{z}\,dx\,dy$$

$$\therefore \int_S (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{s} = -\int_S dx\,dy = -\int_0^2\left[\int_0^{2-x} dy\right]dx$$

$$= -\int_0^2 (2-x)dx = -\left[2x-\frac{x^2}{2}\right]_0^2 = \underline{-2}$$

3) Verify the Stokes's theorem by using the results of 1) and 2).

From a) and b),

$$\int_S (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{s} = \oint \boldsymbol{A}\cdot d\boldsymbol{l}$$

4) Can $\boldsymbol{A}$ be expressed as the gradient of a scalar? Explan.

From a vector null identity, $\nabla\times\nabla V = \boldsymbol{0}$ for any scalar function V,

we can express $\boldsymbol{A} = \nabla V$ if $\nabla\times\boldsymbol{A} = \boldsymbol{0}$.

However, $\nabla\times\boldsymbol{A} = -\hat{z} \neq \boldsymbol{0}$ in this case.

Therefore, $\boldsymbol{A}$ cannot be expressed as ∇V