Plasma Electrodynamics I May 21, 2009

Selected Solutions of Mid-term Exam 2

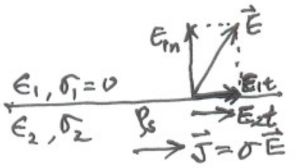
1\. 1) ① $\oint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\epsilon}$: The total outward flux of E-field over any closed surface in a dielectric is equal to the total charge enclosed in the surface divided by ϵ. (Gauss's law)

② $\oint_C \vec{E}\cdot d\vec{\ell} = 0$: The line integral of the static electric field intensity around any closed path vanishes

② is a path-independent integral (only end-point dependent) which vanishes $\Rightarrow$ E is a conservative field.

② is derived from the surface integral of $\nabla\times\vec{E}=\vec{0}$ by using Stokes's theorem. $\nabla\times\vec{E}=\vec{0}$ satisfies a null vector identity $\nabla\times\nabla V=\vec{0}$ if $\vec{E}=-\nabla V$. Therefore, $\vec{E}$ can be obtained from a scalar potential V.

2)



From $\nabla\times\vec{E}=\vec{0}$ and $\vec{J}=\sigma\vec{E}$,

$$E_{1t}=E_{2t}=J_{2t}/\sigma_2 \qquad J_{1t}=0$$

From $\nabla\cdot\vec{D}=\rho_v$,

$$D_{1n}=\rho_s \quad \text{or} \quad E_{1n}=\rho_s/\epsilon_1$$

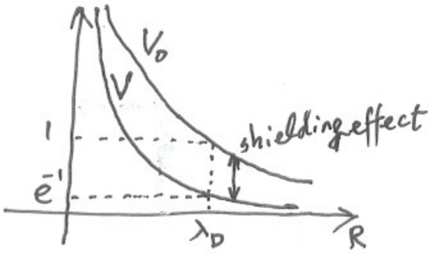
3) For a point charge in free space,

$$V_0(R)=q/4\pi\epsilon_0 R$$

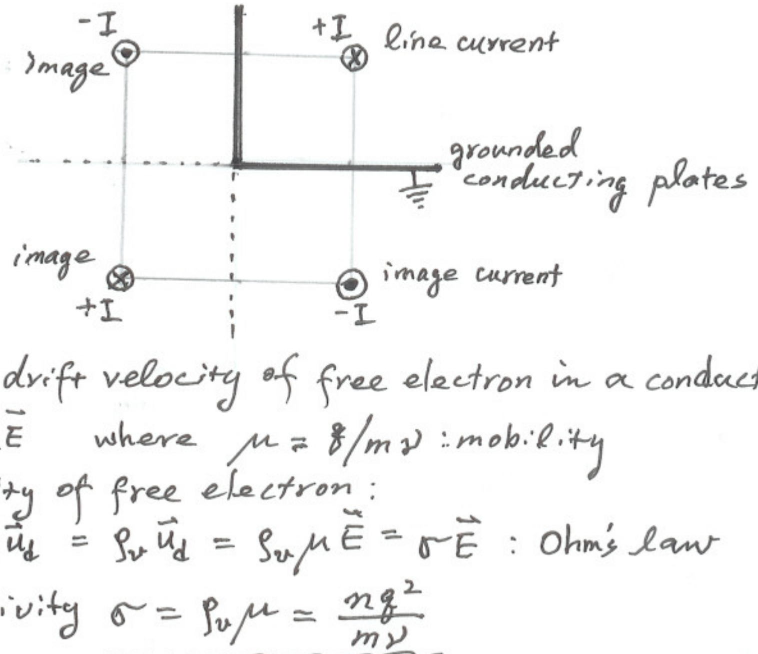
For a test charge in a plasma,

$$V(R)=V_0\, e^{-R/\lambda_D}$$

where λ_D is the Debye length due to the shielding effect of the electron cloud around the test charge.



4)



5) Steady-state drift velocity of free electron in a conductor:

$$\vec{u}_d=\mu\vec{E} \quad \text{where } \mu=q/m\nu \text{ : mobility}$$

Current density of free electron:

$$\vec{J}=nq\vec{u}_d=\rho_v\vec{u}_d=\rho_v\mu\vec{E}=\sigma\vec{E} \text{ : Ohm's law}$$

$\therefore$ conductivity $\sigma=\rho_v\mu=\dfrac{nq^2}{m\nu}$

Both μ and σ are inversely proportional to collision freq. ν.

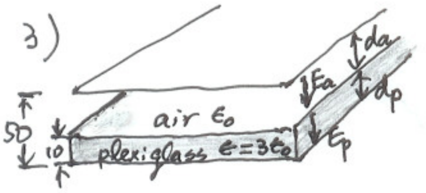
2. Breakdown voltage of a medium with a thickness d

$V_b = E_b d$ where E_b is the dielectric serength.

1) For air, $V_b = E_{ba} d_a = (3\times10^6)(50\times10^{-3}) = \underline{1.5\times10^5 (V) = 150 (kV)}$

2) For plexiglass, $V_b = E_{bp} d_p = (2\times10^7)(50\times10^{-3}) = \underline{10^6 (V) = 1000 (kV)}$

3)



$V_b = E_{ba} d_a + E_{bp} d_p$

Since $D_a = D_p \Rightarrow \epsilon_0 E_a = \epsilon E_p$

$\Rightarrow E_a = \frac{\epsilon}{\epsilon_0} E_p = 3 E_p$

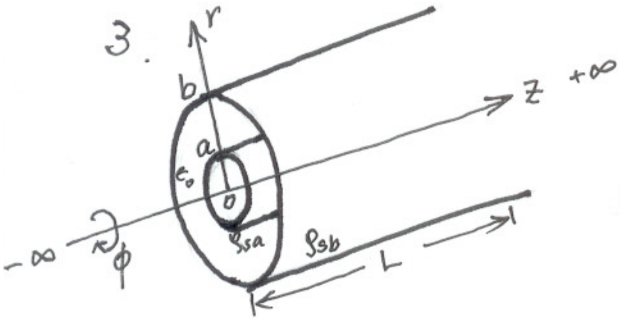
$\Rightarrow E_a > E_p$ (High electric field intensity is applied in air, which means the breakdown occurs in air first.)

Therefore,

$$V_b = E_{ba} d_a + \frac{E_{ba}}{3} d_p = E_{ba}\left(d_a + \frac{d_p}{3}\right)$$

$$= 3\times10^6\left(40\times10^{-3} + \frac{10\times10^{-3}}{3}\right) = \underline{1.3\times10^5 (V) = 130 (kV)}$$

3.



Cylindrical symmetric $\left(\frac{\partial}{\partial\phi} = 0\right)$:

$$\vec{E} = \hat{r} E_r$$

1) applying Gauss's law, $\oint \vec{E}\cdot d\vec{s} = Q/\epsilon_0$

For $r<a$, $\oint E_r ds_r = 0$

$$\Rightarrow \int_0^L \int_0^{2\pi} E_r r d\phi dz = 0$$

$$\Rightarrow 2\pi r L E_r(r) = 0$$

$$\Rightarrow \underline{E_r(r) = 0}$$

For $a<r<b$, $\oint E_r ds_r = Q/\epsilon_0$

$$\Rightarrow \int_0^L \int_0^{2\pi} E_r r d\phi dz = \frac{2\pi a L \rho_{sa}}{\epsilon_0}$$

$$\Rightarrow \cancel{2\pi} r \cancel{L} E_r(r) = \cancel{2\pi} a \cancel{L} \rho_{sa}/\epsilon_0$$

$$\Rightarrow \underline{E_r(r) = \frac{a\rho_{sa}}{\epsilon_0 r}}$$

For $r>b$, $\oint E_r ds_r = Q/\epsilon_0$

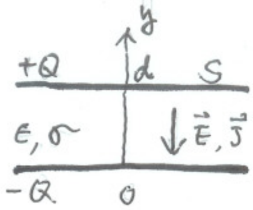
Similarly, $2\pi r L E_r(r) = \frac{2\pi a L \rho_{sa}}{\epsilon_0} + \frac{2\pi b L \rho_{sb}}{\epsilon_0}$

$$\Rightarrow \underline{E_r(r) = \frac{a\rho_{sa} + b\rho_{sb}}{\epsilon_0 r}}$$

2) $E_r(r) = 0$ for $r>0$,

$$\Rightarrow a\rho_{sa} + b\rho_{sb} = 0 \Rightarrow \underline{\frac{b}{a} = -\frac{\rho_{sa}}{\rho_{sb}}}$$

4.



1) $C = Q/V$

$V = -\int_0^d \vec{E}\cdot d\vec{l} = \int_0^d E\,dy = Ed$

$D_n = \rho_s \Rightarrow \epsilon E = \frac{Q}{S} \Rightarrow E = Q/\epsilon S$

$\therefore\ C = Q/Ed = Q/(\cancel{Q}d/\epsilon S) = \epsilon\frac{S}{d}$

2) $W_e = \int_{V'} w_e\,dv = \int_{V'}\left(\frac{\epsilon}{2}E^2\right)dv = \frac{1}{2}\int_{V'}\epsilon\left(\frac{V}{d}\right)^2 dv$

$= \frac{1}{2}\epsilon\left(\frac{V}{d}\right)(Sd) = \frac{1}{2}\left(\epsilon\frac{S}{d}\right)V^2 = \frac{1}{2}CV^2$

3) $C = \frac{Q}{V} = \frac{\oint_S \epsilon\vec{E}\cdot d\vec{s}}{-\int_d \vec{E}\cdot d\vec{l}}$

$R = \frac{V}{I} = \frac{-\int_d \vec{E}\cdot d\vec{l}}{\oint \sigma\vec{E}\cdot d\vec{s}}$

$\Rightarrow RC = \frac{\epsilon\oint_S \vec{E}\cdot d\vec{s}}{\sigma\oint_S \vec{E}\cdot d\vec{s}} = \frac{\epsilon}{\sigma}$

$\therefore\ R = \frac{\epsilon/\sigma}{C} = \frac{\epsilon/\sigma}{\epsilon S/d} = \frac{1}{\sigma}\frac{d}{S} = \eta\frac{d}{S}$

5. 1) BVP: $\nabla^2 V = 0$ in spherical symmetric fields $\left(\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} = 0\right)$

$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = 0,\ R_i \le R \le R_o$ —— ①

BCs: $V(R)|_{R=R_i} = V_0$ —— ②

$V(R)|_{R=R_o} = 0$ —— ③

General solution of ① : $\frac{\partial V}{\partial R} = \frac{C_1}{R^2} \rightarrow V = -\frac{C_1}{R} + C_2$ —— ④

② in ④ :

$V_0 = -C_1/R_i + C_2$
$0 = -C_1/R_o + C_2$
$\Rightarrow C_1 = V_0/\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$, $C_2 = C_1/R_o$
$\Rightarrow V(R) = \frac{V_0}{\frac{1}{R_o} - \frac{1}{R_i}}\left(\frac{1}{R_o} - \frac{1}{R}\right)$

2) $\vec{J} = \sigma\vec{E}$: Ohm's law and $\vec{E} = -\nabla V$

$\vec{E}(R) = \hat{R}(-\nabla V) = -\hat{R}\frac{\partial V}{\partial R} = \hat{R}\frac{V_0}{\frac{1}{R_i} - \frac{1}{R_o}}\frac{1}{R^2}$

$\therefore\ \vec{J}(R) = \sigma\vec{E}(R) = \hat{R}\frac{\sigma V_0}{\frac{1}{R_i} - \frac{1}{R_o}}\frac{1}{R^2}$

3) $R = V_0/I$

$I = \int_S \vec{J}\cdot d\vec{s}_R = \int_0^{2\pi}\int_0^{\pi} J_R R^2\sin\theta\,d\theta\,d\phi = \frac{\sigma V_0}{\frac{1}{R_i} - \frac{1}{R_o}}\underbrace{\int_0^{2\pi}d\phi}_{=2\pi}\underbrace{\int_0^{\pi}\sin\theta\,d\theta}_{=-\cos\theta|_0^{\pi} = 2}$

$= \frac{4\pi\sigma V_0}{\frac{1}{R_i} - \frac{1}{R_o}}$

$\therefore\ R = V_0/I = \frac{1}{4\pi\sigma}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$