Linear Algebra (Eng. Math. 3) Midterm Exam. 2006 Fall

1. ($\pm$30pts) For the following questions, you just need to answer by TRUE or FALSE, or by a short answer. NOTE THAT the correct answer gets the point, but the wrong answer will get MINUS points!

   (a) ($\pm$3pts) A subset of a linearly dependent vector set is linearly dependent. (T/F)

   (b) ($\pm$3pts) If the columns of AB are linearly independent, then so are the columns of B. (T/F)

   (c) ($\pm$3pts) When P is a projection matrix, $(I-P^T)$ is also a projection matrix. (T/F)

   (d) ($\pm$3pts) If $\dim(V) = n$, then any spanning set consisting of n vectors in V is a basis of V. (T/F)

   (e) ($\pm$3pts) If S_1 and S_2 are subsets of a vector space V such that $S_1 \not\subset S_2$, then $\text{span}(S_1) \not\subset \text{span}(S_2)$. (T/F)

   (f) ($\pm$3pts) If there is a solution x to $Ax = b$, then the vector b must be perpendicular to every vector y in $\mathcal{N}(A^T)$. (T/F)

   (g) ($\pm$3pts) If $\det A = 0$ then at least one of the cofactors must be zero. (T/F)

   (h) ($\pm$3pts) If $A = A^2$, then $\det(A)$ is either 0 or 1. (T/F)

   (i) ($\pm$3pts) Consider a 4×4 matrix

   $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

   Which term in the following appears in the calculation of $\det(A)$? Choose as many as possible.

   $$A : a_{13}a_{24}a_{33}a_{42}, \qquad B : a_{41}a_{12}a_{33}a_{34},$$
   $$C : a_{11}a_{21}a_{31}a_{41}, \qquad D : a_{14}a_{23}a_{31}a_{42}.$$

   (j) ($\pm$3pts) Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank of A?

2. (10pts) Let A and b be given by

   $$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

   Find p and q such that $b = p + q$, with p in the column space of A and q perpendicular to that space. Also, answer which of the four fundamental spaces of A contains q.

3. (10pts) Let W be the subspace of $\mathbb{R}^4$ spanned by

$$w_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \qquad w_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 5 \end{bmatrix}.$$

Find a basis for $W^{\perp}$.

4. (10pts) The space of all 2×2 matrices has the four basis "vectors"

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the linear transformation of *transposing* every 2 by 2 matrix, and find the matrix A representing this transformation with respect to the above basis.

5. (10pts) Factor

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

into QR. (QR-Decomposition)

6. (10pts) For an $n \times n$ square matrix S, the 'trace' is defined by

$$\text{trace}(S) = \sum_{i=1}^{n} S_{i,i}.$$

For two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$, prove or disprove that $\text{trace}(AB) = \text{trace}(BA)$.

7. (10pts) Prove or disprove that a linear transformation $T(x) = Ax$ is one-to-one if $\mathcal{N}(A) = \{0\}$.

8. (10pts) Suppose that P is a projection matrix onto the subspace

$$\mathcal{S} = \text{span}\left\{ \begin{bmatrix} 16.07 \\ \log(27) \\ -4 \\ \sqrt{13} \end{bmatrix}, \begin{bmatrix} 2 \\ 15 \\ 0 \\ \sinh(6) \end{bmatrix} \right\},$$

and Q is the projection matrix onto the orthogonal complement $\mathcal{S}^{\perp}$. What are $P + Q$ and PQ?

"Trust what you know."