

#1. (a) FALSE

Example. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

↓
choose two. Then, they are indep.

(b) TRUE

Because, if col. of B are dep., then col. of AB are dep.
 $\exists x \neq 0$ s.t. $Bx=0$ \longrightarrow $\exists x \neq 0$ s.t. $ABx=0$

(c) TRUE.

Since $p^2=p \Rightarrow p^T p^T = p^T$.

check $\{ I - p^T : \text{symmetric} \}$

$$(I - p^T)(I - p^T) = I - p^T - p^T + p^T p^T = I - p^T. \quad \text{OK.}$$

(d) TRUE

Basis : minimum number of any spanning vectors.

(e) FALSE

Example. $S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$S_1 \not\subset S_2$.

But $\text{span } S_1 = \mathbb{R}^2 = \text{span } S_2$

$\therefore \text{span } S_1 \subset \text{span } S_2$.

(f) TRUE.

If $Ax=b$ holds for some x ,

then $y^T Ax = y^T b = 0 \quad \forall y^T A = 0$, i.e., $y \in N(A^T)$.

(g) FALSE.

Example. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(h) TRUE

$$\det A = \det A \cdot \det A \Rightarrow \det A = 0, \text{ or } 1$$

(i) D

Non-zero terms in the determinant comes from the selection of a_{ij} that is not in the same row & column as the other a_{ij} 's.

(j) n.

Since the only sol. is $x=0$, $m \geq n$.

And, the null space has zero dimension.

So, rank $A = n$.

#2

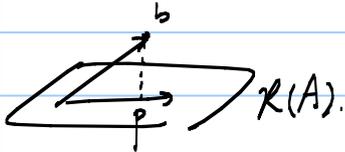
$$b = p + q.$$

$$p \in R(A)$$

$$q \in N(A^T)$$

column space of A

left-null space of A .



$$A\alpha = p$$

$$(b-p)^T A = 0.$$

$$A^T b = A^T A \alpha$$

$\therefore p = A (A^T A)^{-1} A^T b$ if A has full col. rank.

$$A^T A = 9 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

$$\therefore q = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

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#3, $\dim(W^\perp) = 1$. Let w be a basis for W^\perp

$$w^T w_1 = 0 \quad w^T w_2 = 0 \quad w^T w_3 = 0$$

$$\therefore \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} w = 0, \quad \begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & -2 & 3 \\ 0 & -1 & -1 & 5 \end{bmatrix} w = 0$$

Gaussian elimination or else

Ans: $w = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. or its constant multiplication.

#4. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This was your homework.

#5. $a_1 = \begin{pmatrix} c\theta \\ s\theta \end{pmatrix}, \quad a_2 = \begin{pmatrix} s\theta \\ c\theta \end{pmatrix}$

$$q_1 = \frac{a_1}{\|a_1\|} = a_1 = \begin{pmatrix} c\theta \\ s\theta \end{pmatrix}$$

$$a_2' = a_2 - (a_2^T q_1) q_1 = \begin{pmatrix} s\theta \\ c\theta \end{pmatrix} - (s\theta c\theta) \begin{pmatrix} c\theta \\ s\theta \end{pmatrix}$$

$$= \begin{pmatrix} s\theta - s\theta c^2\theta \\ -s^2\theta c\theta \end{pmatrix} = s^2\theta \begin{pmatrix} s\theta \\ -c\theta \end{pmatrix}, \quad \|a_2'\| = \sqrt{s^2\theta - 2s^2\theta c^2\theta + s^2\theta c^4\theta + s^4\theta c^2\theta}$$

$$= s^2\theta \begin{pmatrix} s\theta \\ -c\theta \end{pmatrix} \rightarrow = s^2\theta$$

$$q_2 = \begin{pmatrix} s\theta \\ -c\theta \end{pmatrix}$$

$$Q = [q_1 \ q_2], \quad R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ 0 & a_2^T q_2 \end{bmatrix} = \begin{bmatrix} 1 & s\theta c\theta \\ 0 & s^2\theta \end{bmatrix}$$

* QR decomposition is not unique.

$$\Rightarrow Q = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}, \quad R = \begin{bmatrix} 1 & s\theta c\theta \\ 0 & -s^2\theta \end{bmatrix}$$

is also a sol.

#6. (proof)

$$(AB)_{(i,i)} = \sum_{k=1}^n a_{ik} b_{ki}$$

$$(BA)_{(j,j)} = \sum_{k=1}^m b_{jk} a_{kj}$$

$$\text{trace}(AB) = \sum_{i=1}^m \left(\sum_{k=1}^n a_{ik} b_{ki} \right)$$

$$\text{trace}(BA) = \sum_{j=1}^n \left(\sum_{k=1}^m b_{jk} a_{kj} \right)$$

In fact, both are the same.

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#7. If it is NOT one-to-one,
 $\Rightarrow x_1 \neq x_2$ s.t.

$$Ax_1 = Ax_2.$$

$$\Rightarrow A(x_1 - x_2) = 0.$$

$$\Rightarrow \text{Since } \mathcal{N}(A) = \{0\}, \quad x_1 = x_2.$$

Contradiction !!

\therefore The claim is proved.

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#8. $P + Q = I$

$$PQ = 0 \text{ zero matrix.}$$

(why?).

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