Linear Algebra (Eng. Math. 3) Final Exam. 2006 Fall

1. ($\pm$15pts) Answer by TRUE or FALSE for the following questions with a very short proof or explanation, or a counterexample. NOTE THAT the correct answer gets the point, but the wrong answer will get MINUS points.

   (a) ($\pm$3pts) There is no square matrix A such that A and $A+I$ are similar to each other. (T/F)

   (b) ($\pm$3pts) Singular matrices are not diagonalizable. (T/F)

   (c) ($\pm$3pts) Sum of two positive definite matrices is positive definite. (T/F)

   (d) ($\pm$3pts) If A is skew-symmetric, then e^A is orthogonal. (T/F)

   (e) ($\pm$3pts) A square matrix A is invertible if and only if $A^T A$ is invertible. (T/F)

2. (10pts) Using the singular value decomposition, find the pseudoinverse $A^\dagger$ of A given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

3. (a) (10pts) Find a positive definite symmetric square root, $\sqrt{M}$, of M given by

$$M = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

   (b) (10pts) Let

$$F(x) = \frac{2x_1^2 + 2x_2^2}{5x_1^2 + 8x_1x_2 + 5x_2^2},$$

   which is defined for all non-zero $x = (x_1, x_2)$. Find the minimum value of $F(x)$.

4. (10pts) Let a mapping T is an identity transformation; that is, $T(x) = x$ for all $x \in \mathbb{R}^2$. Find the corresponding matrix A, if the basis for the domain is $v_1 = (1,2)$, $v_2 = (3,4)$ and the basis for the image is $w_1 = (1,0)$ and $w_2 = (0,1)$. (What you are doing is changing the basis.)

5. (10pts) By solving the following difference equation

$$\begin{aligned} x(k+1) &= 0.9x(k) + 0.2y(k), \qquad & x(0) &= 0.5, \\ y(k+1) &= 0.1x(k) + 0.8y(k), \qquad & y(0) &= 0.5, \end{aligned}$$

   find $x(k)$ and $y(k)$. What are $x(\infty)$ and $y(\infty)$?

6. (a) (10pts) Find the characteristic polynomial, $\det(sI - A)$, of the following $A \in \mathbb{R}^{n\times n}$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}.$$

(b) (10pts) Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & -b_3 & \cdots & -b_{n-1} \end{bmatrix}.$$

Find the characteristic polynomial of

$$\begin{bmatrix} A - 2B & 2B \\ -B & A + B \end{bmatrix}.$$

7. (15pts) I wanted to find the Jordan form of

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which has multiple eigenvalues of 1. So, I tried to find eigenvectors, but only two are obtained as

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

With the two, can you find the Jordan matrix J and the nonsingular matrix P such that $P^{-1}AP = J$?