Linear Algebra (Eng. Math. 3) Midterm Exam.
2006 Fall

Questions 1–7 have 10 points each.

1. Find all values of k for which
$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$$
is invertible.

2. Find four fundamental subspaces (column, null, row, and left-null spaces) of a matrix
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}.$$
Express the vector $x = [7, 8]^T$ as the sum of two vectors that belong to two of the four fundamental subspaces respectively.

3. Get a projection matrix P that projects a vector onto the column space of
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

4. Let $A = [3, 1, -1]$, and let V be the nullspace of A. Find a basis for V and a basis for $V^{\perp}$. Find an orthonormal basis for V if your basis for V is not orthonormal.

5. Find all solutions $x = (x_1, x_2, x_3, x_4)$ satisfying the following linear equations
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + x_2 &= 1 \\ x_2 + x_3 &= 0 \\ x_3 + x_4 &= 0 \\ x_1 + x_4 &= 1. \end{aligned}$$
Also, write down every solution x that consists of -1, 0 or 1 as its entry.

6. Consider a linear transformation T, from a set of cubic polynomial P_3 to a set of fourth-degree polynomials P_4, that is a multiplication of $(2+3t)$ to a cubic polynomial. Taking the basis $\{1, t, t^2, t^3\}$ for P_3 and $\{1, t, t^2, t^3, t^4\}$ for P_4, find a matrix A that represents the transformation T.

7. If you know all 16 cofactors of a 4×4 invertible matrix A, how would you find A? (Hint: First of all, can you find $\det A$?)

8. (30pts) Answer by TRUE or FALSE for each of the following problems. You will get 3 points for each question if you write the correct answer with a correct explanation (which can be very short). No answer will get 0 point, but we will get -3 points if you give a wrong answer.

   (a) (T/F) If A is invertible then $A^T A$ is invertible.

   (b) (T/F) Let T be a one-to-one linear transformation from a vector space V to another W. If the vectors $v_1, \cdots, v_k$ in V are linearly independent, then $T(v_1), \cdots, T(v_k)$ are also linearly independent in W.

   (c) (T/F) If $\dim(V) = n$, then any spanning set consisting of n vectors in V is a basis of V.

   (d) (T/F) If S_1 and S_2 are subsets of a vector space V such that $S_1 \not\subset S_2$, then $\text{span}(S_1) \not\subset \text{span}(S_2)$.

   (e) (T/F) Suppose that a subspace V has a basis $\{v_1, v_2, v_3\}$. If a vector $w \neq 0$ is orthogonal to V, the vectors of $\{w, v_1, v_2, v_3\}$ are linearly independent.

   (f) (T/F) If the columns of AB are linearly independent, then so are the columns of B.

   (g) (T/F) If V is orthogonal to W, then $V^\perp$ is orthogonal to $W^\perp$.

   (h) (T/F) If P_C is a projection matrix that projects a vector onto the column space of A, the projection matrix that projects onto the row space of A is P_C^T.

   (i) (T/F) For any square matrix A, which one is correct?

$$\text{(T)} : \text{nullity}(A) \leq \text{nullity}(A^2) \qquad \text{(F)} : \text{nullity}(A) \geq \text{nullity}(A^2)$$

   (j) (T/F) Which one is the Vandermonde matrix?

$$T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{or} \qquad F = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$