

#8

(a) TRUE.

If ATA is not invertible, $\exists x \neq 0$ st $ATAx = 0$
 $\Rightarrow x^T ATA x = 0 \Rightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0$
 $\Rightarrow A$ is not invertible.

~~✗~~

(b) TRUE.

$c_1 T(v_1) + c_2 T(v_2) + \dots + c_k T(v_k) = 0$
 $\Rightarrow T(c_1 v_1 + \dots + c_k v_k) = 0$ ↓ linearity of T
 $\Rightarrow c_1 v_1 + \dots + c_k v_k = 0$ ↓ one-to-one
 $\therefore c_1 = \dots = c_k = 0$.

(c) TRUE. Basis = minimal number of spanning set
n

(d) FALSE. Example $V = \mathbb{R}$. $S_1 = [1]$. $S_2 = [2]$.
 $S_1 \not\subseteq S_2$
 $\text{span } S_1 = \mathbb{R} = \text{span } S_2$. ~~✗~~

(e) TRUE.

$c_0 w + c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$
 $\xrightarrow{w^T x} \Rightarrow c_0 \|w\|^2 = 0 \Rightarrow c_0 = 0$.
 $\Rightarrow c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

(1) TRUE. If not, $\exists x \neq 0$ s.t. $Bx=0$.
 $\Rightarrow ABx=0$. \Rightarrow columns of AB are dependent. ~~✗~~

(2) FALSE. Example: In \mathbb{R}^3 , $V = \text{span}\{e_1\}$, $W = \text{span}\{e_2\}$.
 $\Rightarrow V \perp W$.
 $\Rightarrow V^\perp = \text{span}\{e_2, e_3\}$, $W^\perp = \text{span}\{e_1, e_3\}$.
 $\Rightarrow V^\perp \not\perp W^\perp$ ($\because e_3 \in V^\perp$ and $e_3 \notin W^\perp$).

(3) FALSE. Example: nonsingular A of $n \times n$.

$$P_C = A(A^T A)^{-1} A^T \quad n \times n$$

$$P_C^T = A(A^T A)^{-1} A^T \quad (\because P_C = P_C^T, \text{ s.s.})$$

But, projection to row space of $A = A^T(AA^T)^{-1}A$ ~~✗~~

(i) T. $\text{rank}(A) \geq \text{rank}(A^2)$
 $\therefore \text{nullity}(A) \leq \text{nullity}(A^2)$. ~~✗~~

(j) T. Read the textbook.

#3

$$R(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

$$\Rightarrow \text{Let } \bar{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow (b-p) \perp \bar{A}$$

$$\Rightarrow \bar{A}^T (b - \alpha \bar{A}) = 0, \quad \alpha \bar{A}^T \bar{A} = \bar{A}^T b, \quad \alpha = (\bar{A}^T \bar{A})^{-1} \bar{A}^T b$$

$$\therefore p = \underbrace{\bar{A} (\bar{A}^T \bar{A})^{-1} \bar{A}^T}_{P} b$$

P : projection mtrx.

$$\therefore P = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (2)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\text{답}}$$

* 아무 행과 열이 $P = A(A^T A)^{-1} A^T$ 라는 수동하면 틀리 못함.

#4. Since $A = [3, 1, -1]$, $\dim(N(A)) = 2$

So, I take

$$V = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

basis for V

Since $N(A) \oplus \underbrace{\text{row space of } A}_{V^\perp} = \mathbb{R}^3$,

$$V^\perp = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

basis for V^\perp

New orthonormal basis for V (by Gram-Schmidt process)

$$v_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \leftarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2' = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -3/2 \\ 3 - 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3/2 \\ 3/2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \|v_2'\| = \sqrt{1 + \frac{9}{4} + \frac{9}{4}} = \sqrt{\frac{22}{4}} = \frac{\sqrt{22}}{2}$$

$$\therefore v_2 = \frac{2}{\sqrt{22}} \begin{bmatrix} 1 \\ -3/2 \\ 3/2 \end{bmatrix}$$

$$d_2: \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{22} \\ -3/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix}$$

$\begin{matrix} \parallel \\ v_1 \end{matrix}$
 $\begin{matrix} \parallel \\ v_2 \end{matrix}$

* $\|v_1\| = \|v_2\| = 1$ in L^2 , $v_1^T v_2 = 0$, $A^T v_1 = 0$, $A^T v_2 = 0$ $\Leftrightarrow v_1, v_2$ are orthogonal.

#5. I use Gaussian elimination,

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

x_1 x_2 x_3 x_4 b
pivot free
variable variable

$$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \Leftrightarrow \text{all solution}$$

\uparrow
 particular sol

$\therefore -1, 0, 1 \in \mathbb{Z}$ are the solutions $c = 0, 1$.

$$d_2: X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\#2. \quad R(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$$

$$\begin{cases} N(A) = \{0\} \\ R(A^T) : \text{row space} = \mathbb{R}^2 \end{cases}$$

$$N(A^T) : \text{left-null space} = \text{span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\text{my note} \quad \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \left(\begin{array}{c} -1 \\ 4/3 \\ 1 \end{array} \right)$$

$$x = \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \underset{\text{row space}}{\begin{bmatrix} 1 \\ 8 \end{bmatrix}} + \underset{\text{nullspace}}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

~~4/3~~

$$\#1. \quad A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix} \rightarrow \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ 0 & k-8 & k-4 \end{bmatrix}$$

$$\begin{aligned} \det A &= k \left\{ (k+1)(k-4) - (k-8) \right\} \\ &= k \left(k^2 - 3k - 4 - k + 8 \right) = k \left(k^2 - 4k + 4 \right) \end{aligned}$$

$$\therefore k \neq 0, 2, \cancel{2}$$

∴

#7. $A^{-1} = \frac{1}{\det A} A_{\text{cof}}$ $\circ |P_3$, $\det A \neq 0$ $A \in \mathbb{R}^{4 \times 4}$.

\uparrow
known to us

$\Rightarrow \det(A^{-1}) = \frac{1}{\det A} = \det\left(\frac{1}{\det A} A_{\text{cof}}\right) = \left(\frac{1}{\det A}\right)^4 \det(A_{\text{cof}})$

\swarrow
since $A_{\text{cof}} \in \mathbb{R}^{4 \times 4}$, why? \therefore

$\therefore (\det A)^3 = \frac{1}{\det A_{\text{cof}}}$: now known to us.

$\therefore A = (\det A) A_{\text{cof}}^{-1}$

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invertible since A is invertible.

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#6. $P_3 \xrightarrow{T: (2+3t)} P_4$

$\{1, t, t^2, t^3\}$ $\{1, t, t^2, t^3, t^4\}$

$T(1) = 2+3t$

$T(t) = 2t+3t^2$

$T(t^2) = 2t^2+3t^3$

$T(t^3) = 2t^3+3t^4$

$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix} \in \mathbb{R}^{5 \times 4}$

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cb.

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