Linear Algebra (Eng. Math. 3) Final Exam. 2007 Spring

1. (10pts) Suppose a symmetric matrix $A \in \mathbb{R}^{3\times 3}$ has eigenvectors u, v, and w corresponding to its eigenvalues 0, 1, and 2, respectively. In term of u, v, w, describe the nullspace, left nullspace, row space, and column space of A.

2. (10pts) Prove that all the eigenvalues for a skew-Hermitian matrix are pure imaginary.

3. (10pts) Prove that $\det(e^A) = e^{\text{trace}(A)}$ where $A \in \mathbb{R}^{n\times n}$ has n independent eigenvectors.

4. (10pts) Suppose that $A^T P + PA = -I$ with a positive definite matrix P and a real matrix A. Prove that all the eigenvalues of A have negative real part.

5. (10pts) For a matrix
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix},$$
find its norm $\|A\|$, and a vector x such that $\|Ax\| = \|A\|\|x\|$.

6. (10pts) Consider a function $F(x, y) = (x^2 - 2x)\cos y$. State the type of the point $(x, y) = (1, \pi)$ among a minimum, a maximum, or a saddle.

7. (10pts) Solve the solution of the following differential equation
$$\frac{du}{dt}(t) = Au(t), \qquad u(0) = u_0,$$
where
$$A = \begin{bmatrix} -1 & -2 & 1 \\ 1 & -4 & 1 \\ 1 & -2 & -1 \end{bmatrix},$$
by finding e^{At}. Note that A has three eigenvalues, all of which are -2.

8. (30pts) Answer by TRUE or FALSE for each of the following problems. You will get 3 points for each question if you write the correct answer with a correct explanation (which can be very short). No answer will get 0 point, but you will get -3 points if you give a wrong answer.

   (a) (T/F) If $M^{-1}HM = -H$ for an invertible M, then H has symmetric eigenvalues (that is, if λ is an eigenvalue of H then $-\lambda$ is also an eigenvalue of H).
   (b) (T/F) If A and B are diagonalizable, so is AB.
   (c) (T/F) If eigenvectors x and y correspond to distinct eigenvalues, then $x^H y = 0$.
   (d) (T/F) A and $A + I$ cannot be similar for every matrix A.
   (e) (T/F) When A is invertible, A and A^{-1} have the same eigenvectors.
   (f) (T/F) Hermitian, skew-Hermitian, and unitary matrices are normal matrices.
   (g) (T/F) If λ is any eigenvalue of A, then $|\lambda| \leq \|A\|$.
   (h) (T/F) For any matrix $A \in \mathbb{R}^{m\times n}$, $\|A\| = \|A^T\|$.
   (i) (T/F) The only positive definite projection matrix is $P = I$.
   (j) (T/F) The pseudoinverse of 2 by 2 zero matrix is 2 by 2 zero matrix.

9. (Problem for Repeat Takers; 10pts) For a real symmetric matrix A and a positive definite matrix P, prove that PA is diagonalizable and all eigenvalues are real.