

#1. (Prob. 5.5.14) A : symmetric \Rightarrow u, v, w orthogonal

$$\text{rank}(A) = 2$$

$$\therefore \text{nullity}(A) = 1.$$

$$Au = 0 \cdot u = 0 \quad \Rightarrow \quad \therefore u \in N(A)$$

$$Av = 1 \cdot v \quad \Rightarrow \quad \therefore v \in R(A)$$

$$Aw = 2 \cdot w \quad \Rightarrow \quad \therefore w \in R(A)$$

$$\therefore N(A) = \text{span}\{u\} \quad \Rightarrow \quad \therefore R(A^T) = \text{span}\{v, w\}$$

$$\therefore R(A) = \text{span}\{v, w\} \quad \Rightarrow \quad \therefore N(A^T) = \text{span}\{u\}.$$

#2 Let $Ax = \lambda x$

\uparrow skew-Hermitian

$$\Rightarrow x^H Ax = \lambda \|x\|^2 \quad \Rightarrow \quad (x^H Ax)^H = (\lambda \|x\|^2)^H$$

$$\| \quad = \bar{\lambda} \|x\|^2$$

$$x^H A^H x = -x^H Ax$$

$$= -\lambda \|x\|^2$$

$$\therefore (\lambda + \bar{\lambda}) \|x\|^2 = 0, \quad \|x\| \neq 0$$

$$\therefore \lambda + \bar{\lambda} = 0$$

$$\Rightarrow \text{Re}(\lambda) = 0$$

$$\Rightarrow \lambda: \text{pure imaginary}$$

#3. $A = S \Lambda S^{-1}$

$$\therefore e^A = S e^\Lambda S^{-1}$$

$$\therefore \det(e^A) = \det S \det e^\Lambda \det S^{-1}$$

$$= \det e^\Lambda$$

$$= e^{\lambda_1} \cdot e^{\lambda_2} \cdot e^{\lambda_3} \dots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \dots + \lambda_n}$$

$$= e^{\text{trace}(A)}$$

#4. Let $Ax = \lambda x$.

$$x^H (A^T P + P A) x = - \|x\|^2$$

$$= \bar{\lambda} x^H P x + \lambda x^H P x = - \|x\|^2$$

$$\therefore (\bar{\lambda} + \lambda) \underset{\underset{0}{\downarrow}}{x^H P x} = - \underset{\underset{0}{\downarrow}}{\|x\|^2}$$

$$\therefore \bar{\lambda} + \lambda = 2 \text{Re}[\lambda] < 0$$

#5. Find the maximum singular value of A

$$A^T A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \text{e.v} \Rightarrow (\lambda - 2)(\lambda - 8) - 16 = \lambda^2 - 10\lambda = 0$$

$$\therefore \lambda = [2, 10, 0]$$

$$\sigma_{\max} = \sqrt{10} = \|A\| \rightarrow$$

Find the e-vector x for $A^T A$ with $\lambda = 10$.



$$\rightarrow A^T A - 10I = \begin{bmatrix} -8 & 4 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad (\text{이 상수배})$$

#6.

$$\bar{x} = x - 1$$

$$\bar{y} = y - \pi$$

$$\begin{aligned} F(\bar{x}+1, \bar{y}+\pi) &= ((\bar{x}+1)^2 - 2(\bar{x}+1)) \cos(\bar{y}+\pi) \\ &= (\bar{x}^2 + 2\bar{x} + 1 - 2\bar{x} - 2) (-\cos \bar{y}) \\ &= -(\bar{x}^2 - 1) \cos \bar{y} \end{aligned}$$

$$\begin{bmatrix} \frac{\partial^2 F}{\partial \bar{x} \partial \bar{x}} & \frac{\partial^2 F}{\partial \bar{x} \partial \bar{y}} \\ \frac{\partial^2 F}{\partial \bar{x} \partial \bar{y}} & \frac{\partial^2 F}{\partial \bar{y} \partial \bar{y}} \end{bmatrix}_{(0,0)} = \begin{bmatrix} -2 \cos 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

negative definite.

\therefore Maximum point.

*참조:

$$\text{Jacobian of } F = \begin{bmatrix} (2x-2) \cos y \\ -(x^2-2x) \sin y \end{bmatrix}$$

$$\begin{aligned} \text{Hessian of } F &= \begin{bmatrix} 2 \cos y & -(2x-2) \sin y \\ -(2x-2) \sin y & -(x^2-2x) \cos y \end{bmatrix} \quad (x,y) = (1, \pi) \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

#8.

(a) T : H and $-H$ are similar so have the same e.v.

If $\det(\lambda I - H) = 0$, then $\det(-\lambda I + H) = 0$.
So, if λ is e.v of H , then $-\lambda$ is e.v of $-H$.

(b) F : Let $A = \begin{bmatrix} \frac{1}{2} & \Delta \\ 0 & \frac{1}{3} \end{bmatrix}$, $B = \begin{bmatrix} 2 & \Delta \\ 0 & 3 \end{bmatrix}$

distinct e.v \Rightarrow diagonalizable

$AB = \begin{bmatrix} 1 & \frac{\Delta}{2} + 3\Delta \\ 0 & 1 \end{bmatrix}$. If $\frac{\Delta}{2} + 3\Delta = 1$,
then AB is NOT diagonalizable

(c) F : but x and y are independent.

(d) T : since e.v of $A+I$ is greater than A by 1,
they cannot be similar.

(e) T : $Ax = \lambda x \Leftrightarrow \frac{1}{\lambda}x = A^{-1}x$
 $\lambda \neq 0$.

(f) T : $NN^H = N^H N$ hold for all. $H^H = H$ or $H^H = -H$ or $H^H = H^{-1}$

(g) T : let $Ax = \lambda x$ and $\|x\|=1$

$\|A\| = \max_{\|y\|=1} \|Ay\| \geq \|Ax\| = |\lambda| \|x\| = |\lambda|$

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(b) $T : \|A\| = \sqrt{\lambda_{\max}(A^T A)}$ \searrow the same.
 $\|A^T\| = \sqrt{\lambda_{\max}(A A^T)}$

(i) $T : P^2 = P \Rightarrow P(P - I) = 0$
 $P \text{ p.d.} \Rightarrow P \text{ invertible} \Rightarrow P - I = 0$.

(ii) $T : \begin{bmatrix} \circ & \circ \\ \circ & \circ \\ \vdots & \vdots \end{bmatrix} = V \underset{\text{diag}}{\Sigma} U^T \Rightarrow \Sigma^T = \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$

#9. P is p.d $\Rightarrow \sqrt{P}$ is well-defined.
 $\underset{P^{\frac{1}{2}}}{\sqrt{P}}$ and $(\sqrt{P})^{-1} = P^{-\frac{1}{2}}$ is also well-defined.

$(\because P = Q \Lambda Q^T, P^{-\frac{1}{2}} = Q \underset{\text{diag}}{\Lambda^{-\frac{1}{2}}} Q^T)$

$\begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \frac{1}{\sqrt{\lambda_2}} & \\ & & \ddots \end{pmatrix}$

$PA \leftrightarrow (\sqrt{P})^{-1} (PA) (\sqrt{P}) = \sqrt{P} A \sqrt{P}$

Similar

Symmetric

\therefore this has real e.v.s and is diagonalizable.

$\therefore PA$ has real e.v.s and is diagonalizable

#7. E. vectors, $A - (-2)I = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore \exists$ two indep. e-vectors $x_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

A.M = 3
G.M = 2. $\therefore J = \begin{bmatrix} -2 & 1 \\ & -2 \\ & & -2 \end{bmatrix}$

Generalized e-vector = x_1, x_2, z

$M \triangleq [x_1 \ z \ x_2]$ (or $M \triangleq [x_2 \ z \ x_1]$)

$\Rightarrow AM = MJ$

$\therefore Az = x_1 - 2z \Rightarrow (A+2I)z = x_1$

$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} z = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

\uparrow
No solution ???

 Null space of $(A - (-2)I)$: $\text{span}\{x_1, x_2\}$

$= \left\{ y : y = \alpha_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} z = \alpha_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

find z, α_1, α_2 .

\Rightarrow From the shape of $(A+2I)$, $\alpha_1 = 1, \alpha_2 = 1$



$$\rightarrow z = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ (or any vector s.t. } [1, -2, 1] z = 2 \text{)}$$

$$\rightarrow M = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix} \quad \leftarrow \text{여러가지 가능}$$

답:

$$u(t) = M e^{Jt} M^{-1} u_0$$

$$e^{Jt} = \begin{bmatrix} e^{-2t} & t e^{-2t} & \\ & e^{-2t} & \\ & & e^{-2t} \end{bmatrix}$$

