Midterm Exam (10_2)

1. Suppose that $z = s + v$, where *s* and *v* are independent and $v \sim N(0, \sigma_v^2)$. Further assume that we have a knowledge on *s*, viz., $s \sim N(\overline{s}, \sigma_s^2)$.
   (1) Find the likelihood function and the maximum likelihood (ML) estimate $\hat{s}_{ML}$. (5 points)
   (2) Find the maximum *a posteriori* (MAP) estimate $\hat{s}_{MAP}$. (10 points)
   (3) Find the mean square (MS) estimate $\hat{s}_{MS}$. (15 points)

2. A stochastic unknown *X* and the data *Z* are jointly distributed according to

$$f_{XZ}(x,z) = \frac{6}{7}(x+z)^2; \quad 0 \le x \le 1, 0 \le z \le 1.$$

   We want to find the best linear mean square (LMS) estimate $\hat{X}_{LMS}(z)$.

   (1) First, calculate the best estimate for *X* and its error covariance in the absence of data *Z* to compare with the LMS estimate. (5 points)

   (2) Show that $\hat{X}_{LMS}(z)$ is obtained from

$$\hat{X}_{LMS}(z) = \overline{X} + P_{XZ} P_Z^{-1}(z - \overline{z}). \quad \text{(10 points)}$$

   (3) Find $\hat{X}_{LMS}(z)$ using the equation in (2). (10 points)

   (4) Calculate the error covariance of $\hat{X}_{LMS}(z)$ and compare with the result in (1). (5 points)

3. Suppose a signal $s(n)$ is generated by the stochastic difference equation

$$s(n) = 0.75 s(n-1) + w(n)$$

   where $w(n)$ is white noise $w(n) \sim N(0, 4.9)$. The signal is measured in the presence of additive white noise $v(n) \sim N(0, 8)$ that is uncorrelated with the signal.
   (1) Show that the power spectral density of $s(n)$ is given by

$$S_s(z) = \frac{4.9}{(1 - 0.75z^{-1})(1 - 0.75z)}. \quad \text{(5 points)}$$

   (2) Find the causal Wiener filter for estimating $\hat{s}(n)$ from the noisy

measurements. Give the filter's transfer function $H(z)$ and impulse response $h(n)$. (15 points)

(3) Show that the mean square estimation error (MSE) can be written by

$$\text{MSE} = R_{\text{s}}(0) - \sum_{i=0}^{\infty} h(i) R_{\text{sz}}(i)\ .\ \text{(5 points)}$$

(4) Using the equation in (3), compute the MSE associated with $\hat{s}(n)$. (5 points)