Midterm Exam (10_2)

1. Suppose that $z = s + v$, where *s* and *v* are independent and $v \sim N(0, \sigma_v^2)$. Further assume that we have a knowledge on *s*, viz., $s \sim N(\overline{s}, \sigma_s^2)$.
   (1) Find the likelihood function and the maximum likelihood (ML) estimate $\hat{s}_{ML}$. (5 points)
   (2) Find the maximum *a posteriori* (MAP) estimate $\hat{s}_{MAP}$. (10 points)
   (3) Find the mean square (MS) estimate $\hat{s}_{MS}$. (15 points)

2. A stochastic unknown *X* and the data *Z* are jointly distributed according to

$$f_{XZ}(x,z) = \frac{6}{7}(x+z)^2; \quad 0 \le x \le 1, 0 \le z \le 1.$$

   We want to find the best linear mean square (LMS) estimate $\hat{X}_{LMS}(z)$.

   (1) First, calculate the best estimate for *X* and its error covariance in the absence of data *Z* to compare with the LMS estimate. (5 points)

   (2) Show that $\hat{X}_{LMS}(z)$ is obtained from

$$\hat{X}_{LMS}(z) = \overline{X} + P_{XZ}P_Z^{-1}(z - \overline{z}). \quad \text{(10 points)}$$

   (3) Find $\hat{X}_{LMS}(z)$ using the equation in (2). (10 points)

   (4) Calculate the error covariance of $\hat{X}_{LMS}(z)$ and compare with the result in (1). (5 points)

3. Suppose a signal $s(n)$ is generated by the stochastic difference equation

$$s(n) = 0.75s(n-1) + w(n)$$

   where $w(n)$ is white noise $w(n) \sim N(0, 4.9)$. The signal is measured in the presence of additive white noise $v(n) \sim N(0, 8)$ that is uncorrelated with the signal.
   (1) Show that the power spectral density of $s(n)$ is given by

$$S_s(z) = \frac{4.9}{(1 - 0.75z^{-1})(1 - 0.75z)}. \quad \text{(5 points)}$$

   (2) Find the causal Wiener filter for estimating $\hat{s}(n)$ from the noisy

measurements. Give the filter's transfer function $H(z)$ and impulse response $h(n)$. (15 points)

(3) Show that the mean square estimation error (MSE) can be written by

$$\text{MSE} = R_s(0) - \sum_{i=0}^{\infty} h(i) R_{sz}(i) \text{ . (5 points)}$$

(4) Using the equation in (3), compute the MSE associated with $\hat{s}(n)$. (5 points)

(Solutions)

1.

(1) (Kamen Example 3.3)

The likelihood function is

$$f_{z|s}(z \mid s) = f_v(z-s) = \frac{1}{\sqrt{2\pi}\sigma_v} e^{-(z-s)^2/2\sigma_v^2}.$$

$$\hat{s}_{ML} = z.$$

(2) (Kamen Example 3.6)

$$\widehat{s}_{MAP} = \overline{s} + \frac{\sigma_s^2}{\sigma_v^2 + \sigma_s^2}(z - \overline{s}) \quad or \ \hat{s}_{MAP} = \frac{\sigma_v^2 \overline{s} + \sigma_s^2 z}{\sigma_v^2 + \sigma_s^2}.$$

(3)(Kamen Example 3.8)

$$\widehat{s}_{MS} = \hat{s}_{MAP}.$$

2.

(1)

$$f_X(x) = \int_{-\infty}^{\infty} f_{XZ}(x,z)dz = \int_0^1 \frac{6}{7}(x+z)^2\, dz = \frac{6}{7}\left(x^2 + x + \frac{1}{3}\right)$$

$$\hat{X}_{ap} = \overline{X} = \frac{9}{14} = 0.643$$

$$P_{\tilde{X}_{ap}} = E\left\{\tilde{X}_{ap}^2\right\} = E\left\{\left(X - \frac{9}{14}\right)^2\right\} = 0.0677.$$

(2) (ETNote10_2.doc)

(3)

$$P_{XZ} = E\{XZ\} - \overline{X}\overline{Z} = \int_0^1 \int_0^1 xz \frac{6}{7}(x+z)^2\, dxdz - \overline{X}\overline{Z} = -0.0085$$

$$P_Z = E\left\{Z^2\right\} - \overline{Z}^2 = \int_0^1 z^2 \frac{6}{7}\left(z^2 + z + \frac{1}{3}\right)dz - \overline{Z}^2 = 0.068$$

$$\hat{X}_{LMS}(z) = 0.643 - \frac{0.0085}{0.068}(z - 0.643) = 0.724 - 0.126z.$$

(4)

$$
\begin{aligned}
P_{\tilde{X}_{LMS}} &= E\left\{\left[\left(X-\overline{X}\right)-P_{XZ}P_Z^{-1}\left(Z-\overline{Z}\right)\right]\left[\left(X-\overline{X}\right)-P_{XZ}P_Z^{-1}\left(Z-\overline{Z}\right)\right]^T\right\} \\
&= E\left\{\left(X-\overline{X}\right)\left(X-\overline{X}\right)^T-P_{XZ}P_Z^{-1}\left(Z-\overline{Z}\right)\left(X-\overline{X}\right)^T\right. \\
&\qquad \left.-\left(X-\overline{X}\right)\left(Z-\overline{Z}\right)^T P_Z^{-1}P_{ZX}+P_{XZ}P_Z^{-1}\left(Z-\overline{Z}\right)\left(Z-\overline{Z}\right)^T P_Z^{-1}P_{ZX}\right\} \\
&= P_{\tilde{X}}-P_{XZ}P_Z^{-1}P_{ZX}=P_{\tilde{X}}-P_{XZ}P_Z^{-1}P_{XZ}^T \le P_{\tilde{X}}
\end{aligned}
$$

In this problem,

$$P_{\tilde{X}} \approx 0.0677$$
$$P_{\tilde{X}_{LMS}} \approx 0.0666.$$

3. (Kamen Problem 4.11)

(1)

$$s(n) = 0.75s(n-1) + w(n)$$

$$S_s(z) = 0.75z^{-1}S_s(z) + S_w(z);\ S_w(z) = 4.9$$

$$H_s(z) = \frac{S_s(z)}{S_w(z)} = \frac{1}{1-0.75z^{-1}}$$

$$S_s(z) = H_s(z)H_s(z^{-1})S_w(z) = \frac{4.9}{\left(1-0.75z^{-1}\right)(1-0.75z)}$$

$$R_s(n) = 11.2(0.75)^{|n|}$$

(2)

$$S_z(z) = S_s(z) + S_v(z) = \frac{4.9}{\left(1-0.75z^{-1}\right)(1-0.75z)} + 8$$

$$= \frac{15\,(1-0.4z)\left(1-0.4z^{-1}\right)}{\left(1-0.75z^{-1}\right)(1-0.75z)}$$

$$S_z^+(z) = \frac{3.873\left(1-0.4z^{-1}\right)}{\left(1-0.75z^{-1}\right)};\quad S_z^-(z) = \frac{3.873\,(1-0.4z)}{(1-0.75z)}$$

$$\frac{S_{sz}(z)}{S_z^-(z)} = \frac{(1-0.75z)}{3.873\,(1-0.4z)} \cdot \frac{4.9}{\left(1-0.75z^{-1}\right)(1-0.75z)}$$

$$= \frac{1.265}{(1-0.4z)\left(1-0.75z^{-1}\right)} = \frac{-3.16z^{-1}}{\left(1-0.75z^{-1}\right)\left(1-2.5z^{-1}\right)}$$

$$= \frac{1.81}{1-0.75z^{-1}} - \frac{1.81}{1-2.88z^{-1}}$$

$$\left[\frac{S_{sz}(z)}{S_z^-(z)}\right]_+ = \frac{1.81}{1-0.75z^{-1}}$$

$$H(z) = \frac{1}{S_z^+(z)}\left[\frac{S_{sz}(z)}{S_z^-(z)}\right]_+$$

$$= \frac{\left(1-0.75z^{-1}\right)}{3.873\left(1-0.4z^{-1}\right)} \cdot \frac{1.81}{\left(1-0.75z^{-1}\right)} = \frac{0.4667}{\left(1-0.4z^{-1}\right)}$$

$$h(n) = 0.4667(0.4)^n 1(n).$$

(3) (ETNote10_3.doc)

(4)

$$
\begin{aligned}
\mathrm{MSE} &= R_{\mathrm{s}}(0)-\sum_{i=0}^{\infty} h(i) R_{\mathrm{sz}}(i) \\
&=11.2-\sum_{i=0}^{\infty} 0.4667 \times 0.4^{i} \times 11.2 \times 0.75^{i} \\
&=11.2-5.227 \sum_{i=0}^{\infty} 0.4^{i} \times 0.75^{i} \\
&=11.2-5.227 \times \frac{1}{1-0.4 \times 0.75} \approx 3.733
\end{aligned}
$$