Estimation Theory Final Test

December 8, 2010

1. A simplified spacecraft tracking problem is formulated by

$$\begin{aligned} \dot{x}_c &= w_c, \qquad w_c \sim N(0,q) \\ z_c &= x_c + v_c, \quad v_c \sim N(0,r). \end{aligned} \qquad \text{(T10.2.1-1)}$$

   (a) Suppose that the measurements are taken every 0.5 second. Show that the discrete model for Eq. (T10.2.1-1) is given by

$$\begin{aligned} x(k+1) &= x(k) + w(k) \\ z(k) &= x(k) + v(k). \end{aligned} \qquad \text{(T10.2.1-2)}$$

   Determine the mean and variance of $w(k)$ and $v(k)$.

   (b) Find the steady-state Kalman filter solution for this problem assuming that $w(k)$ and $v(k)$ are white Gaussian and uncorrelated each other.

   (c) Find the causal Wiener filter solution for this problem and compare it with the result of (b).

   (d) Find the steady-state optimal smooth solution for this problem and compare it with the result of (b).

   (For (b), (c), and (d), let $q = r = 1$.)

2. Consider the following state and measurement equations

$$x_{k+1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} x_k + w_k, \quad w_k \sim \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}\right), \quad x_0 \sim (0, P_0)$$

$$z_k = Hx_k + v_k, \quad v_k \sim (0,1).$$

(a) Determine if the steady-state Kalman gain K is asymptotically stable when $H = \begin{bmatrix} 0 & 3 \end{bmatrix}$.

(b) What if $H = \begin{bmatrix} 3 & 3 \end{bmatrix}$?

3. The equation of motion of a pendulum hanging from a ceiling is given by the differential equation

$$\frac{d^2\theta(t)}{dt^2} + \sin\theta(t) = w(t),$$

where $\theta(t)$ is the angular position of the pendulum at time *t*. Discrete measurements of $\theta(t)$ are given by $z(n) = \theta(n) + v(n)$, where the sampling interval $T = 0.5$. Suppose that $w(t) \sim N(0, 0.02)$, $v(n) \sim N(0,1)$, and $\theta(0) \sim N(0, 0.02)$. Give the equations for the EKF that provides an estimate of $\theta(i)$, $i = 1,2,\cdots$, based on the measurements $z(i)$ for $i = 1,2,\cdots,n$.

4. Suppose that RV *x* is uniformly distributed on [-1, 1], and $y = e^{2x}$.

(a) What is the mean of *y*, $\bar{y}$?

(b) What is the first-order approximation to $\bar{y}$?

(c) What is the second-order approximation to $\bar{y}$?

(d) What is the unscented approximation to $\bar{y}$?

(e) What is the variance of *y*? What is the unscented approximation to the variance of *y*?