Estimation Theory Final Test_Solution

December 8, 2010

1. A simplified spacecraft tracking problem is formulated by

$$\dot{x}_c = w_c, \qquad w_c \sim N(0,q)$$
$$z_c = x_c + v_c, \quad v_c \sim N(0,r). \qquad \text{(T10.2.1-1)}$$

(a) Suppose that the measurements are taken every 0.5 second. Show that the discrete model for Eq. (T10.2.1-1) is given by

$$x(k+1) = x(k) + w(k)$$
$$z(k) = x(k) + v(k). \qquad \text{(T10.2.1-2)}$$

Determine the mean and variance of $w(k)$ and $v(k)$.

(Solution) (10 points)

Given

$$\dot{x} = w \equiv A_c x + G_c w, \quad A_c = 0, G_c = 1, w \sim N(0,q)$$
$$z = x + v, \quad v \sim N(0,r)$$

the discrete model is given by

$$x(k+1) = e^{A_c T} x(k) + w(k) = x(k) + w(k) \equiv Ax(k) + Gw(k) \qquad \text{(T2.1-1)}$$
$$z(k) = x(k) + v(k) \equiv Hx(k) + v(k) \qquad \text{(T2.1-2)}$$

where

$$w(k) \sim N\left(0, \int_0^T e^{A_c \tau} G_c Q G_c^T \left(e^{A_c \tau}\right)^T d\tau = Tq = 0.5q\right)$$
$$v(k) \sim N\left(0, \frac{r}{T} = 2r\right).$$

(b) Find the steady-state Kalman filter solution for this problem assuming that $w(k)$ and $v(k)$ are white Gaussian and uncorrelated each other.

(Solutiom) (20 points)

Find P_∞ by solving the ARE

$$P_\infty = A\left[P_\infty - P_\infty H^T\left(HP_\infty H^T + R\right)^{-1} HP_\infty\right]A^T + GQG^T$$

$$= \left[P_\infty - P_\infty\left(P_\infty + 2r\right)^{-1} P_\infty\right] + \frac{q}{2}$$

$$= P_\infty - \frac{P_\infty^2}{P_\infty + 2r} + \frac{q}{2}$$

$$\frac{P_\infty^2}{P_\infty + 2r} = \frac{q}{2}$$

$$P_\infty^2 - \frac{q}{2}P_\infty - rq = 0$$

$$P_\infty = \frac{q}{4} \pm \sqrt{\left(\frac{q}{4}\right)^2 + rq}.$$

Since $P_\infty > 0$,

$$P_\infty = \frac{q}{4} + \sqrt{\left(\frac{q}{4}\right)^2 + rq} = \frac{1}{4}\left(q + \sqrt{q^2 + 16rq}\right). \qquad \text{(T2.1-3)}$$

And

$$K_\infty = P_\infty H\left(HP_\infty H^T + R\right)^{-1}$$

$$= P_\infty\left(P_\infty + 2r\right)^{-1}$$

$$= \frac{P_\infty}{P_\infty + 2r} \qquad \text{(T2.1-4)}$$

$$= \frac{\frac{1}{4}\left(q + \sqrt{q^2 + 16rq}\right)}{\frac{1}{4}\left(q + \sqrt{q^2 + 16rq}\right) + 2r}.$$

The transfer function for the steady-state Kalman filter, $H_{SSKF}(z)$, is

$$H_{SSKF}(z) = H\left(zI - A\left[I - K_\infty H\right]\right)^{-1} AK_\infty = \frac{K_\infty}{z - 1 + K_\infty} \qquad \text{(T2.1-5)}$$

Let $q = r = 1$ for comparison,

$$H_{SSKF}(z) \approx \frac{0.39}{z - 0.61}. \qquad \text{(T2.1-6)}$$

(c) Find the causal Wiener filter solution for this problem and compare it with the result of (b).

(Solution) (20 points)

Suppose $w(k)$ and $v(k)$ are white, then

$S_W(z) = 0.5q$

From $x(k+1) = x(k) + w(k)$,

$$H_{wto\,x}(z) = \frac{S_X(z)}{S_W(z)} = \frac{1}{Z^{-1}-1} \qquad \text{(T2.1-7)}$$

And

$$\begin{aligned} S_X(z) &= S_W(z) H_{wto\,x}(z) H_{wto\,x}(z^{-1}) \\ &= 0.5q \cdot \frac{1}{z^{-1}-1} \cdot \frac{1}{z-1} \qquad \text{(T2.1-8)} \\ &= \frac{0.5q}{(1-z^{-1})(1-z)} \end{aligned}$$

Let,

$$s(k) = x(k)$$
$$z(k) = s(k) + v(k)$$

Then,

$$\begin{aligned} S_S(z) &= S_X(z) \\ S_Z(z) &= S_S(z) + S_V(z) \\ &= \frac{0.5q}{(1-z)(1-z^{-1})} + 2r \qquad \text{(T2.1-9)} \end{aligned}$$

$$S_{SZ}(z) = S_S(z) = \frac{0.5q}{(1-z)(1-z^{-1})}$$

$$H_{Wiener}(z^{-1}) = \frac{1}{S_Z^{+}(z)} \left[\frac{S_{SZ}(z)}{S_Z^{-1}(z)} \right]_{+} \qquad \text{(T2.1-10)}$$

(Note: Refer to Eq.(T2.1-7))

To compare with (b), let q=r=1, then

$$\begin{aligned} S_Z(z) &= \frac{0.5}{(1-z)(1-z^{-1})} + 2 \\ &= \frac{0.5 + 2(1-z)(1-z^{-1})}{(1-z)(1-z^{-1})} \\ &= \frac{4.5 - 2z - 2z^{-1}}{(1-z)(1-z^{-1})} \\ &= \frac{1.22(1.64-z)(1.64-z^{-1})}{(1-z)(1-z^{-1})} \\ &= \frac{1.10(1.64-z^{-1})}{(1-z^{-1})} \cdot \frac{1.10(1.64-z)}{(1-z)} \\ &\equiv S_Z^{+}(z) S_Z^{-}(z) \end{aligned}$$

$$\frac{S_{SZ}(z)}{S_Z^{-1}(z)} = \frac{0.5}{(1-z)(1-z^{-1})} \cdot \frac{(1-z)}{1.10(1.64-z)}$$

$$= \frac{0.45}{(1-z^{-1})(1.64-z)}$$

$$= \frac{0.70}{(1-z^{-1})} - \frac{0.70}{(1-1.64Z^{-1})}$$

$$\left[\frac{S_{SZ}(z)}{S_Z^{-1}(z)}\right]_+ = \frac{0.70}{1-z^{-1}}$$

$$H_{Wiener}(z^{-1}) = \frac{(1-z^{-1})}{1.10(1.64-z^{-1})} \cdot \frac{0.70}{(1-z^{-1})}$$

$$= \frac{0.39}{1-0.61z^{-1}}$$

$$H_{Wiener}(z) = \frac{0.39}{z-0.61} \qquad \text{(T2.1-11)}$$

Comparing Eqs. (T2.1-6) and (T2.1-11), we see that $H_{SSKF}(z) = H_{Wiener}(z)$.

(d) Find the steady-state optimal smooth solution for this problem and compare it with the result of (b).

(For (b), (c), and (d), let $q = r = 1$.)

(Solution)

The forward filter is the same as (b), viz,

$$P_\infty = \frac{1}{4}(q + \sqrt{q^2 + 16rq})$$
$$K_\infty = \frac{P_\infty}{P_\infty + 2r} \qquad \text{(T2.1-12)}$$

The backward filter is given by

$$S_\infty = A^T\left[I - S_\infty G\left[G^T S_\infty G + Q^{-1}\right]^{-1} G^T\right] S_\infty A + H^T R^{-1} H$$
$$= \left[1 - S_\infty\left(S_\infty + \frac{1}{0.5q}\right)^{-1}\right] S_\infty + \frac{1}{2r} \qquad \text{(T2.1-13)}$$

Solve Eq.(T2.1-13) for S_∞ to obtain

$$S_\infty = \frac{1}{4r}\left[1 + \sqrt{1 + \frac{16}{q}}\right] \qquad (S_\infty > 0)$$

The steady-state optimal smoother is obtained by

$$K_S = P_\infty S_\infty\left[I + P_\infty S_\infty\right]^{-1}$$
$$P_S = \left[I - K_S\right] P_\infty$$

For a simple comparison, let q=r=1. Then,

$$P_\infty = \frac{1}{4}\left(1+\sqrt{17}\right) = 1.28$$
$$S_\infty = \frac{1}{4}\left(1+\sqrt{17}\right) = 1.28$$
$$K_S = 1.28 \times 1.28 \times \left[1 + 1.28 \times 1.28\right]^{-1}$$
$$= 0.62$$
$$P_S = (1-0.62) \times 1.28 = 0.49$$

We can see that the steady-state error variance is reduced to 0.49 from 1.28.

2. Consider the following state and measurement equations

$$x_{k+1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} x_k + w_k, \quad w_k \sim \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}\right), \quad x_0 \sim (0, P_0)$$
$$z_k = Hx_k + v_k, \quad v_k \sim (0,1).$$

(a) Determine if the steady-state Kalman gain K is asymptotically stable when $H = \begin{bmatrix} 0 & 3 \end{bmatrix}$.

(Solution) (10 points)

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = I, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} = \sqrt{Q}\sqrt{Q}^T$$

$$\sqrt{Q} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{0.5} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.707 \end{bmatrix}$$

Reachability test for $\left(A, G\sqrt{Q}\right)$.

$$\rho \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0.707 & 0 & 0.707 & \cdots \end{bmatrix} = 1.$$

Therefore, $\left(A, G\sqrt{Q}\right)$ is non-reachable.

Now, test detectability for $\left(A, H\right)$. When $H = [0 \quad 3]$,

$$A - LH = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -3l_1 \\ 0 & 1-3l_2 \end{bmatrix}.$$

(A, H) is non-detectable since we can not move the eigenvalue 4.

According to Theorem 5-1 and 5-2 in the note, the steady-state error with the Kalman gain K is asymptotically unstable. In addition, it is not guaranteed that for every choice of P_0 there is a bounded limiting solution *P*.

(b) What if $H = [3 \quad 3]$?

(Solution) (10 points)

When $H = [3 \quad 3]$,

$$A - LH = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 - 3l_1 & -3l_1 \\ -3l_2 & 1 - 3l_2 \end{bmatrix}.$$

Two eigenvalues of *A* may be arbitrarily located by properly choosing l_1 and l_2. Therefore, (A, H) is detectable. The steady-state error with the Kalman gain *K* is asymptotically unstable. However, for every choice of P_0 there is a bounded limiting solution *P*.

3. The equation of motion of a pendulum hanging from a ceiling is given by the differential equation

$$\frac{d^2\theta(t)}{dt^2} + \sin\theta(t) = w(t),$$

where $\theta(t)$ is the angular position of the pendulum at time *t*. Discrete measurements of $\theta(t)$ are given by $z(n) = \theta(n) + v(n)$, where the sampling interval $T = 0.5$. Suppose that $w(t) \sim N(0, 0.02)$, $v(n) \sim N(0,1)$, and $\theta(0) \sim N(0, 0.02)$. Give the equations for the EKF that provides an estimate of $\theta(i)$, $i = 1, 2, \cdots$, based on the measurements $z(i)$ for $i = 1, 2, \cdots, n$.

(Solution) (Kamen's Problem 8.8)

4. Suppose that RV *x* is uniformly distributed on [-1, 1], and $y = e^{2x}$.

(a) What is the mean of *y*, $\bar{y}$?

(Solution) (5 points)

$$\bar{y} = E\{y\} = E\{e^x\} = \int_{-1}^{1} \frac{1}{2} e^{2x} dx = \frac{1}{4}\left(e^2 - e^{-2}\right) = 1.813$$

(b) What is the first-order approximation to $\bar{y}$?

(Solution) (5 points)

1st order approximation

$$\bar{y} \simeq h(\bar{x}) + \left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}} E\{\tilde{x}\} = 1.$$

(c) What is the second-order approximation to $\bar{y}$?

(Solution) (5 points)

2nd order approximation

$$\bar{y} = E\{h(x)\} \simeq h(\bar{x}) + E\left\{D_{\tilde{x}}h + \frac{1}{2!}D_{\tilde{x}}^2 h\right\}$$

$$= 1 + \frac{1}{2}\left.\frac{\partial^2 h}{\partial x^2}\right|_{x=\bar{x}} E\left\{\tilde{x}^2\right\} = 1 + \frac{1}{2}\cdot 4P_x = \frac{5}{3} \approx 1.666.$$

(d) What is the unscented approximation to $\bar{y}$?

(Solution) (5 points)

Since

$$\bar{x} = 0, \quad \sigma_x^2 = \frac{1}{3}$$

and

$$\tilde{x}^{(1)} = \frac{1}{\sqrt{3}}, \quad \tilde{x}^{(2)} = -\frac{1}{\sqrt{3}},$$

we obtain

$$\bar{y}_u = \frac{1}{2}\left(e^{\frac{2}{\sqrt{3}}} + e^{-\frac{2}{\sqrt{3}}}\right) = 1.744..$$

(e) What is the variance of y? What is the unscented approximation to the variance of y?

(Solution) (10 points)

$$\text{var}(y) = E\left\{y^2\right\} - \bar{y}^2 = E\left\{e^{4x}\right\} - (1.813)^2 = 6.8225 - 3.2885 = 3.534..$$

$$\text{var}(y_u) = \frac{1}{2}\sum_{i=1}^{2}\left[h\left(\tilde{x}^{(i)}\right) - \bar{y}_u\right]^2 = \frac{1}{2}\left[\left(e^{\frac{2}{\sqrt{3}}} - 1.744\right)^2 + \left(e^{-\frac{2}{\sqrt{3}}} - 1.744\right)^2\right] \approx 2.042.$$