Final Exam		SNU ID #		
Date	2010. 12. 18	Name		
Problem Number				Total
Grader Max				
Grader Score				

1. Euler angle $\boldsymbol{\gamma} = [\phi \quad \theta \quad \psi]^T$ represents the orientation of the body-fixed frame(b-frame) with respect to the inertial reference frame(n-frame).

1.1. Draw the final orientation of the b-frame after it is rotated by using the ZYX-Euler angle $[\phi \quad \theta \quad \psi]^T$, where the angle ψ denotes the first rotational angle about the z-axis of b-frame. After it has undergone the first rotation, the b-frame is denoted as b'-frame. And the b'-frame rotates about the y'-axis of the b'-frame, and the rotational angle of the second rotation is denoted θ. After the second rotation, the b'-frame is denoted as b"-frame. then the b"-frame rotates about the x"-axis of the b"-frame. The rotational angle of the third rotation is denoted ϕ.

1.2. Derive the rotational transformation matrix ${}^{n}\mathbf{R}_{b}$ by using the ZYX-Euler angle.

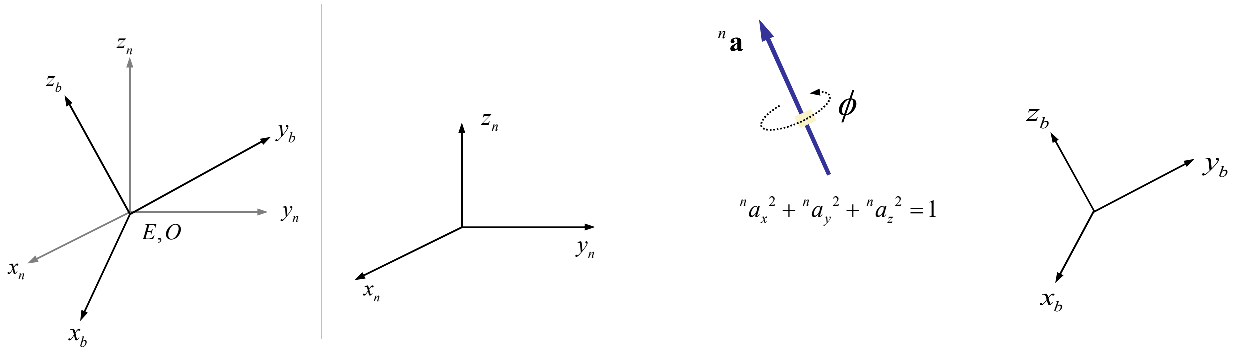
1.3. Explain the relationship between the time derivative of the ZYX-Euler angle $\dot{\boldsymbol{\gamma}} = \begin{bmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^T$ and the angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$. The explanation should include the equation for calculating the angular velocity using the time derivative of ZYX-Euler angle.

1.4. The rotation sequence of the Euler angle should be defined to represent the orientation of certain frame or rigid body. However, if the rotation angles are very small (infinitesimal rotation), the sequence of rotation is commutative. Prove this statement by comparison between the rotational transformation matrix ${}^{n}\mathbf{R}_{b}$ of the ZYX-Euler angle and the rotational transformation matrix ${}^{n}\mathbf{R}_{b}$ of XYZ-Euler angle.

1.5. If the rotation angles are very small (infinitesimal rotation), the angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$ can be considered as the time derivative of the ZYX-Euler angle $\dot{\boldsymbol{\gamma}} = \begin{bmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^T$. Prove this statement by using the answer of the problem 1.3

1.6. Explain the two kinds of gimbal lock, where the ZYX-Euler angle is $\boldsymbol{\gamma} = \begin{bmatrix} \phi & 90^\circ & \psi \end{bmatrix}^T$

2. Euler parameter $\mathbf{p} = [\theta_0 \quad \theta_1 \quad \theta_2 \quad \theta_3]^T$ represents the orientation of the body-fixed frame(b-frame) with respect to the inertial reference frame(n-frame).



$$\mathbf{R}_{\mathbf{a},\phi} = \begin{bmatrix} (1-\cos\phi)\,{}^n a_x^2 + \cos\phi & (1-\cos\phi)\,{}^n a_x\,{}^n a_y - {}^n a_z \sin\phi & (1-\cos\phi)\,{}^n a_z\,{}^n a_x + {}^n a_y \sin\phi \\ (1-\cos\phi)\,{}^n a_x\,{}^n a_y + {}^n a_z \sin\phi & (1-\cos\phi)\,{}^n a_y^2 + \cos\phi & (1-\cos\phi)\,{}^n a_y\,{}^n a_z - {}^n a_x \sin\phi \\ (1-\cos\phi)\,{}^n a_z\,{}^n a_x - {}^n a_y \sin\phi & (1-\cos\phi)\,{}^n a_y\,{}^n a_z + {}^n a_x \sin\phi & (1-\cos\phi)\,{}^n a_z^2 + \cos\phi \end{bmatrix}$$

$${}^n\mathbf{R}_b = \begin{bmatrix} 2(\theta_0{}^2 - \theta_1{}^2) - 1 & 2(\theta_1\theta_2 - \theta_0\theta_3) & 2(\theta_1\theta_3 + \theta_0\theta_2) \\ 2(\theta_1\theta_2 + \theta_0\theta_3) & 2(\theta_0{}^2 + \theta_2{}^2) - 1 & 2(\theta_2\theta_3 - \theta_0\theta_1) \\ 2(\theta_1\theta_3 - \theta_0\theta_2) & 2(\theta_2\theta_3 + \theta_0\theta_1) & 2(\theta_0{}^2 + \theta_3{}^2) - 1 \end{bmatrix}$$, where $\theta_0 = \cos\frac{\phi}{2}$ $, \theta_1 = {}^n a_x \sin\frac{\phi}{2}$ $, \theta_2 = {}^n a_y \sin\frac{\phi}{2}$ $, \theta_3 = {}^n a_z \sin\frac{\phi}{2}$

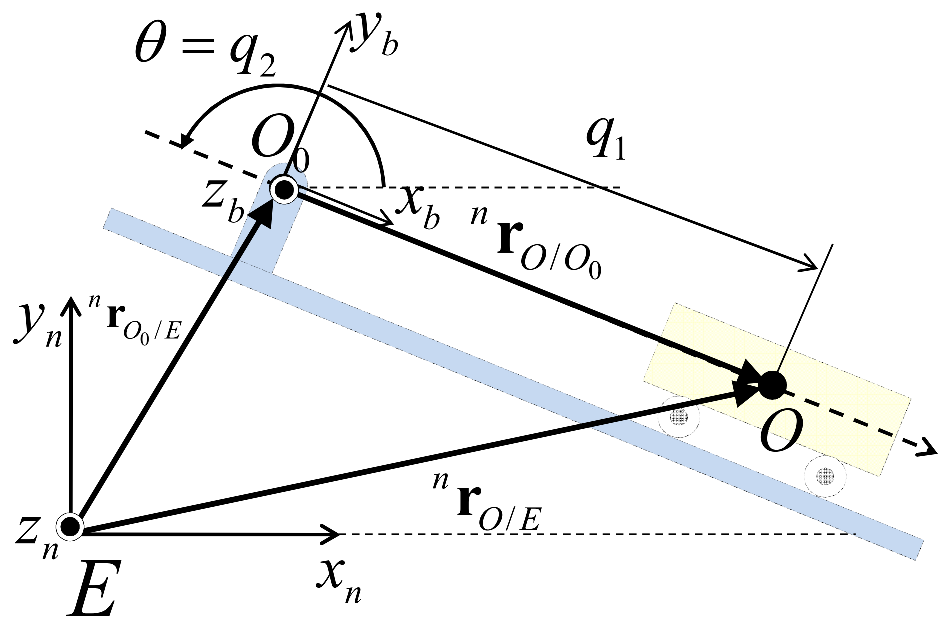
2.1. Derive the rotational transformation matrix ${}^n\mathbf{R}_b$ by using the Euler parameter.

2.2. Explain that how to calculate the Euler parameter $\mathbf{p} = \begin{bmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T$ by using the rotational transformation matrix ${}^{n}\mathbf{R}_{b}$.

2.3. Using the answer of the problem 2.2, explain the reason that why the gimbal lock can be prevented by using the Euler parameter instead of the Euler angle.

2.4. Explain the relationship between the time derivative of the Euler parameter $\dot{\mathbf{p}} = \begin{bmatrix} \dot{\theta}_0 & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix}^T$ and the angular velocity ${}^{n}\boldsymbol{\omega}_{b/n}$. The explanation should include the equation for calculating the angular velocity using the time derivative of Euler parameter.

3. There is a vehicle constrained to move along the straight track. The external force exerted on the vehicle and the track is gravitational force. Answer the following questions.



$n-frame$: Inertial reference frame.
O: Center of mass of the vehicle.
O_0: Center of mass of the track, hinge joint.
${}^{n}\mathbf{r}_{O_0/E}$: Initial position vector of the center of mass O_0 with respect to the point E decomposed in n-frame. ${}^{n}\mathbf{r}_{O_0/E}$ is constant.
${}^{n}\mathbf{r}_{O/E}$: Position vector of the point O with respect to the point E decomposed in n-frame.
${}^{n}\mathbf{r}_{O/O_0}$: Position vector of the point O_0 with respect to the point O decomposed in n-frame.
θ: Angle of inclination of the track.
m_v : Mass of the vehicle $\quad$ m_t : Mass of the track
I_t : Mass moment of inertia of the vehicle about z_b-axis.

3.1. Derive the equations of motion of the vehicle and the track represented by q_1, q_2 using embedding formulation.

$$\boxed{\tilde{\mathbf{M}}\ddot{\mathbf{q}} + \tilde{\mathbf{k}} = {}^{E}\tilde{\mathbf{F}}^{e}}$$

→ Embedding Formulation

$$,\left(\tilde{\mathbf{M}} = \mathbf{J}^T\mathbf{M}\mathbf{J},\ \tilde{\mathbf{k}} = \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{q}},\ {}^{E}\tilde{\mathbf{F}}^{e} = \mathbf{J}^T\ {}^{E}\mathbf{F}^{e}\right)$$

3.2. Show that $\tilde{\mathbf{k}}$ term, which is in the equations of motion derived by embedding formulation, represents Coriolis and centrifugal acceleration using the answer of the problem 3.1.

3.3. Derive the equations of motion of the vehicle and the track represented by $x_{O/E}, y_{O/E}, x_{O_0/E}, y_{O_0/E}, \theta$ using augmented formulation.

$$\begin{bmatrix} \mathbf{M} & \nabla\mathbf{C}^T \\ \nabla\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{O/E} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_O^e \\ 0 \end{bmatrix}$$ → Augmented Formulation

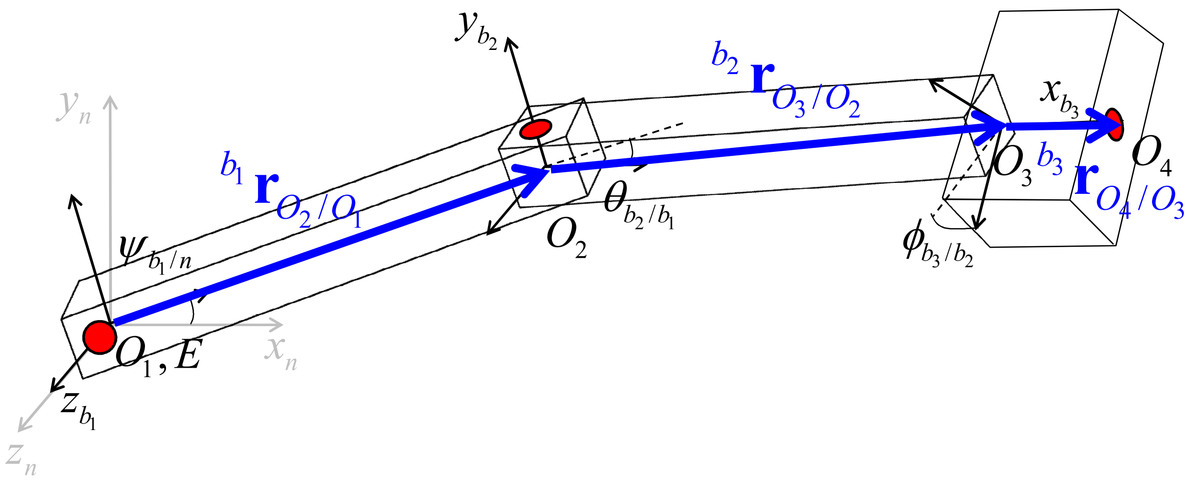
$\mathbf{C} = 0$: Constraint equation

$\mathbf{F}_O^e$: External force

$\boldsymbol{\lambda}$: Lagrange Multiplier

$\mathbf{M}$: Mass matrix

4. There is a 3-link arm moving in the 3-dimensional space. The coordinates of this system are defined in the following figure.



n-frame: Inertial reference frame

b_1-frame: body fixed frame attached to the link 1

b_2-frame: body fixed frame attached to the link 2

b_3-frame: body fixed frame attached to the link 3

$^{b_1}\mathbf{r}_{O_2/O_1}$: Position vector of the point O_2 with respect to the point O_1 decomposed in b_1-frame

$^{b_2}\mathbf{r}_{O_3/O_2}$: Position vector of the point O_3 with respect to the point O_2 decomposed in b_2-frame

$^{b_3}\mathbf{r}_{O_4/O_3}$: Position vector of the point O_4 with respect to the point O_3 decomposed in b_3-frame

$\psi_{b_1/n}$: Rotation angle of the b_1-frame with respect to n-frame about z_{b1}-axis

θ_{b_2/b_1} : Rotation angle of the b_2-frame with respect to b_1-frame about y_{b2}-axis

ϕ_{b_3/b_2} : Rotation angle of the b_3-frame with respect to b_2-frame about x_{b3}-axis

Answer the following questions.

4.1. Calculate the position of the point O_4 with respect to the origin of the n-frame E, which is denoted ${}^{n}\mathbf{r}_{O_4/E}$.

4.2. Calculate the velocity of the point O_4 with respect to the origin of the n-frame E, which is denoted $d\,{}^{n}\mathbf{r}_{O_4/E}\,/\,dt$.

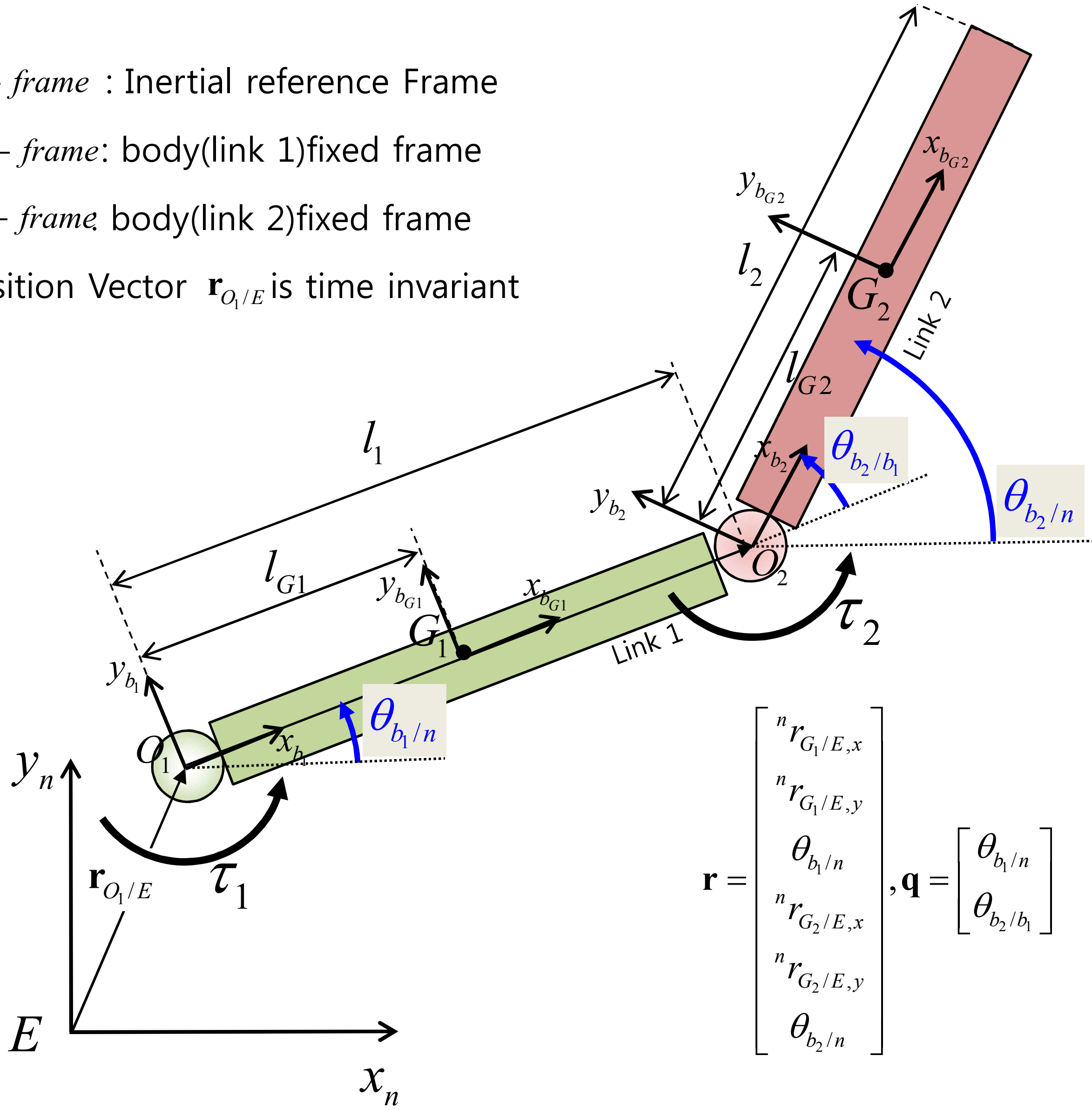
5. There is a 2-link arm moving in the xy-plane. The coordinates of this system are defined in the following figure. Only gravitational force is exerted on this system.

$n-frame$: Inertial reference Frame

$b_1-frame$: body(link 1)fixed frame

$b_2-frame$: body(link 2)fixed frame

Position Vector $\mathbf{r}_{O_1/E}$ is time invariant



$$\mathbf{r}=\begin{bmatrix} {}^n r_{G_1/E,x} \\ {}^n r_{G_1/E,y} \\ \theta_{b_1/n} \\ {}^n r_{G_2/E,x} \\ {}^n r_{G_2/E,y} \\ \theta_{b_2/n} \end{bmatrix}, \mathbf{q}=\begin{bmatrix} \theta_{b_1/n} \\ \theta_{b_2/b_1} \end{bmatrix}$$

Mass and mass moment of inertia about z_{G1}-axis through point G_1 of link 1: m_1, I_1

Mass and mass moment of inertia about z_{G2}-axis through point G_2 of link 2: m_2, I_2

Answer the following questions.

5.1. The equations of motion of the link 2 derived by recursive formulation are as follows. Drive the equation (1) and the equation (3).

Equations of motion for link 2

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1} + \mathbf{S}_{b_2} \cdot \dot{q}_2 \longleftarrow \text{(1) Velocity of } \{b_2\}$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2} \longleftarrow \text{(2) Velocity of } \{b_{G2}\}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \dot{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2 \longleftarrow \text{(3)Acceleration of } \{b_2\}$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2} \longleftarrow \text{(4) Acceleration of } \{b_{G2}\}$$

$${}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^* {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \longleftarrow \text{(5) Force and moment exerted on } \{b_{G2}\}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^*_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}_{G_2}^{B_2} + {}^{b_2}\mathbf{X}^*_{b_3} \cdot {}^{b_3}\hat{\mathbf{f}}_{O_3} \longleftarrow \text{(6) Force and moment exerted on } \{b_2\}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^T \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2} \longleftarrow \text{(7) Torque exerted on the joint of link 2}$$

where

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{O_2/E} \end{bmatrix}, {}^{b_2}\mathbf{X}_{b_1} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & 0 \\ {}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1} \times & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix}, \mathbf{S}_{b_2} = \begin{bmatrix} {}^{b_2}\mathbf{k}_{b_2} \\ 0 \end{bmatrix},$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = \begin{bmatrix} {}^{b_2}\dot{\boldsymbol{\omega}}_{b_2/n} \\ {}^{b_2}\dot{\mathbf{v}}_{O_2/E} \end{bmatrix}, \quad q_2 = \theta_{b_2/b_1}, {}^{b_2}\hat{\mathbf{v}}_{b_2} \times = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{O_2/E} \end{bmatrix} \times = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \times & 0 \\ {}^{b_2}\mathbf{v}_{O_2/E} \times & {}^{b_2}\boldsymbol{\omega}_{b_2/n} \times \end{bmatrix}$$

$${}^{b_2}\hat{\mathbf{v}}_{b_2} \times^* = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \\ {}^{b_2}\mathbf{v}_{O_2/E} \end{bmatrix} \times^* = \begin{bmatrix} {}^{b_2}\boldsymbol{\omega}_{b_2/n} \times & {}^{b_2}\mathbf{v}_{O_2/E} \times \\ 0 & {}^{b_2}\boldsymbol{\omega}_{b_2/n} \times \end{bmatrix}, {}^{b_2}\mathbf{X}^*_{b_1} = \begin{bmatrix} {}^{b_2}\mathbf{R}_{b_1} & -{}^{b_2}\mathbf{R}_{b_1} \cdot {}^{b_1}\mathbf{r}_{O_2/O_1} \times \\ 0 & {}^{b_2}\mathbf{R}_{b_1} \end{bmatrix}, {}^{b_2}\hat{\mathbf{f}}_{O_2} = \begin{bmatrix} {}^{b_2}\mathbf{m}_{O_2} \\ {}^{b_2}\mathbf{f}_{O_2} \end{bmatrix}$$

5.2. The equations of motion of the 2-link arm derived by recursive formulation are as follows.

1) Explain the reason that ${}^{b_1}\hat{\mathbf{v}}_{b_1}, {}^{b_1}\hat{\mathbf{a}}_{b_1}$ are not given, but ${}^{n}\hat{\mathbf{v}}_{n}, {}^{n}\hat{\mathbf{a}}_{n}$ are given in this problem.

2) Explain the reason that ${}^{b_2}\hat{\mathbf{f}}_{O_2}$ is not given, but ${}^{b_3}\hat{\mathbf{f}}_{O_3}$ is given in this problem.

3) Describe the detail sequence for calculating τ_1 and τ_2.

4) Explain the reason that this formulation is called as recursive formulation.

Given: ${}^{b_3}\hat{\mathbf{f}}_{O_3}, {}^{n}\hat{\mathbf{v}}_{n}, {}^{n}\hat{\mathbf{a}}_{n}, q_1, \dot{q}_1, \ddot{q}_1, q_2, \dot{q}_2, \ddot{q}_2, and\ all\ \mathbf{X}, \mathbf{S}, \hat{\mathbf{I}}$ Find: τ_2, τ_2

Equations of motion for link 1

$${}^{b_1}\hat{\mathbf{v}}_{b_1} = {}^{b_1}\mathbf{X}_n \cdot \overset{1)}{{}^{n}\hat{\mathbf{v}}_n} + \mathbf{S}_{b_1} \cdot \dot{q}_1$$

$${}^{b_{G1}}\hat{\mathbf{v}}_{b_{G1}} = {}^{b_{G1}}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{v}}_{b_1}$$

$${}^{b_1}\hat{\mathbf{a}}_{b_1} = {}^{b_1}\mathbf{X}_n \cdot \overset{1)}{{}^{n}\hat{\mathbf{a}}_n} + \mathbf{S}_{b_1} \cdot \ddot{q}_1 + \overset{\circ}{\mathbf{S}}_{b_1} \cdot \dot{q}_1 + {}^{b_1}\hat{\mathbf{v}}_{b_1} \times \mathbf{S}_{b_1} \cdot \dot{q}_1$$

$${}^{b_{G1}}\hat{\mathbf{a}}_{b_{G1}} = {}^{b_{G1}}\mathbf{X}_{b_1} \cdot {}^{b_1}\hat{\mathbf{a}}_{b_1}$$

$${}^{b_{G1}}\hat{\mathbf{f}}^{B}_{G_1} = {}^{b_{G1}}\hat{\mathbf{I}}_{G_1} \cdot {}^{b_{G1}}\hat{\mathbf{a}}_{b_{G1}} + {}^{b_{G1}}\hat{\mathbf{v}}_{b_{G1}} \times^{*} {}^{b_{G1}}\hat{\mathbf{I}}_{G_1} \cdot {}^{b_{G1}}\hat{\mathbf{v}}_{b_{G1}}$$

$${}^{b_1}\hat{\mathbf{f}}_{O_1} = {}^{b_1}\mathbf{X}^{*}_{b_{G1}} \cdot {}^{b_{G1}}\hat{\mathbf{f}}^{B}_{G_1} + {}^{b_1}\mathbf{X}^{*}_{b_2} \cdot \overset{2)}{{}^{b_2}\hat{\mathbf{f}}_{O_2}}$$

$$\tau_1 = \mathbf{S}_{b_1}{}^{T} \cdot {}^{b_1}\hat{\mathbf{f}}_{O_1}$$

Equations of motion for link 2

$${}^{b_2}\hat{\mathbf{v}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot \overset{1)}{{}^{b_1}\hat{\mathbf{v}}_{b_1}} + \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{v}}_{b_2}$$

$${}^{b_2}\hat{\mathbf{a}}_{b_2} = {}^{b_2}\mathbf{X}_{b_1} \cdot \overset{1)}{{}^{b_1}\hat{\mathbf{a}}_{b_1}} + \mathbf{S}_{b_2} \cdot \ddot{q}_2 + \overset{\circ}{\mathbf{S}}_{b_2} \cdot \dot{q}_2 + {}^{b_2}\hat{\mathbf{v}}_{b_2} \times \mathbf{S}_{b_2} \cdot \dot{q}_2$$

$${}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} = {}^{b_{G2}}\mathbf{X}_{b_2} \cdot {}^{b_2}\hat{\mathbf{a}}_{b_2}$$

$${}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2} = {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{a}}_{b_{G2}} + {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}} \times^{*} {}^{b_{G2}}\hat{\mathbf{I}}_{G_2} \cdot {}^{b_{G2}}\hat{\mathbf{v}}_{b_{G2}}$$

$${}^{b_2}\hat{\mathbf{f}}_{O_2} = {}^{b_2}\mathbf{X}^{*}_{b_{G2}} \cdot {}^{b_{G2}}\hat{\mathbf{f}}^{B}_{G_2} + {}^{b_2}\mathbf{X}^{*}_{b_3} \cdot \overset{2)}{{}^{b_3}\hat{\mathbf{f}}_{O_3}}$$

$$\tau_2 = \mathbf{S}_{b_2}{}^{T} \cdot {}^{b_2}\hat{\mathbf{f}}_{O_2}$$