459.666A: The first midterm exam

Answer either in English or in Korean.

Each sub-problem (5pt, unless indicated otherwise) is relatively independent.
Don't panic and give up too soon. Good luck!

1. Boltzmann Response (30pts total)

Statistical mechanics states that the probability distribution function of electrons f(E) in a thermal equilibrium is given as a function of particle's energy "E" and proportional to

$$\exp\left(-\frac{E}{T_e}\right)$$

where T_e is the electron temperature.

Consider an electrostatic fluctuation "φ" in a plasma with a uniform magnetic field in z direction, and $n_0 = n_0(x)$. In the presence of this fluctuation, with the wave complex amplitude proportional to $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$,

a. what condition in terms of the phase velocity of this fluctuation (wave) should be satisfied for the electrons to be considered in a thermal equilibrium? State the condition mathematically, and discuss its physical meaning.

b. From this, derive the Boltzmann response of electron density, n_e is proportional to $\exp(|e|\varphi/T_e)$.

c. Derive the same electron density expression from the following electron momentum equation along the magnetic field.

$$m_e \frac{d}{dt} u_{ez} = -|e| E_z - \frac{1}{n_0} \frac{\partial}{\partial z} P_e$$

d. State the reason why this response is also called "adiabatic response".

e. Obtain the associated electron temperature fluctuation. $\delta T_e / T_{e,c}$.

f. Show that with this adiabatic response, there's no fluctuation-driven electron particle flux in the direction of density gradient up to the second order in fluctuation amplitude, perpendicular to the equilibrium magnetic field, regardless of ion dynamics.

2. Electron Drift Instability for bi-Maxwellian electron distribution

(40pts total)

Consider a drift wave type electrostatic fluctuation "φ" in a collisionless plasma with a uniform magnetic field in z direction, B_0, and $n_0 = n_0(x)$. The wave complex amplitude is proportional to $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. The electron distribution function is given by a bi-Maxwellian characterized by $T_{\parallel}$ for motions along z and $T_{\perp}$ for motions in (x,y) plane. Both temperatures vary in x direction, and

$$F_0 = \left(\frac{m_e}{2\pi}\right)^{3/2} \frac{n_0}{T_\perp T_\parallel^{1/2}} \exp\left(-m_e v_\perp^2/2T_\perp - m_e v_\parallel^2/2T_\parallel\right).$$

In this geometry, the electron drift-kinetic equation is given by

$$\frac{\partial}{\partial t} f_e + c\frac{\vec{E}\times\vec{B}}{B^2}\cdot\vec{\nabla}_\perp f_e + v_z \frac{\partial}{\partial z} f_e - \frac{|e|}{m_e} E_z \frac{\partial}{\partial v_z} f_e = 0, \quad (1)$$

a. State the reason why a term looking like $\frac{|e|}{m_e} E_y \frac{\partial}{\partial v_y} f_e$ or $\frac{|e|}{m_e c} v_y B_0 \frac{\partial}{\partial v_x} f_e$ is absent in Eq. (1).

b. In deriving the linearized electron drift-kinetic equation for $\delta f_e = f_e - F_0$,

$$\frac{\partial}{\partial t}\delta f_e + c\frac{\delta\vec{E}\times\vec{B}}{B^2}\cdot\vec{\nabla}_\perp F_0 + v_z\frac{\partial}{\partial z}\delta f_e - \frac{|e|}{m}\delta E_z \frac{\partial}{\partial v_z} F_0 = 0, \quad (2)$$

some terms have been ignored. Under what conditions they can be ignored? What are their physical meanings?

c. Identify two terms from Eq. (2) which yield the adiabatic electron response in an appropriate limit.

d. **(10pts)** Calculate the dominant imaginary part of perturbed density response from the resonant electron-wave interaction as a correction for the adiabatic electron response. Use an appropriate frequency ordering, and the following mathematical identity.

For $G_0(v) = n_0\left(\frac{m}{2\pi T_0}\right)^{1/2} \exp\left(-\frac{v^2}{2v_{Te}^2}\right)$,

$$\text{Res}\int_{-\infty}^{\infty} \frac{G_0(v)\,dv}{\omega - k_z v} \cong -i\left(\frac{\pi}{2}\right)^{1/2} \frac{n_0}{k_z v_{Te}}$$

e. From the following ion continuity equation, obtain the ion density response .

$$\frac{\partial}{\partial t}\delta n_i + \frac{c\,\delta\vec{E}\times\vec{B}}{B^2}\cdot\vec{\nabla}_\perp n_0 + n_0\,\vec{\nabla}_\perp\cdot\vec{u}_{pol} = 0,$$ here $\vec{u}_{pol}$ is the polarization drift.

f. From the results in "d" and "e", and the quasi-neutrality condition, derive a dispersion relation for an electron drift instability for this system.

g. Discuss the different roles played by $T_{\parallel}$ and by $T_{\perp}$.

3. Ion Temperature Gradient Instability (30pts total)

Consider a low-frequency electrostatic fluctuation "$\delta\phi$" in a collisionless plasma with a uniform magnetic field in z direction, $n_0 = n_0(x)$ and $T_i = T_0(x)$. The wave complex amplitude is proportional to $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. Electrons obey Boltzmann response.

a. State the reasons why ITG in stabilities are of primary research interests in magnetic fusion energy research?

b. Why can ITG instability occur (unstable) even with a Boltzmann electron response?

Consider the following linearized fluid equations for ions. $\delta\vec{u}_E$ is the perturbed ExB drift.

$$\frac{\partial}{\partial t}\delta n_i + \delta\vec{u}_E\cdot\vec{\nabla} n_0 + n_0 \nabla_{\parallel}\delta u_{\parallel} = 0 \quad , \quad (1)$$

$$M_i \frac{\partial}{\partial t}\delta u_{\parallel} = -|e| \nabla_{\parallel}\delta\phi - \frac{1}{n_0}\nabla_{\parallel}\delta P_i \quad , \quad (2)$$

$$\frac{\partial}{\partial t}\delta P_i + \delta\vec{u}_E\cdot\vec{\nabla} P_0 + \Gamma P_0 \nabla_{\parallel}\delta u_{\parallel} = 0 \quad , \quad (3)$$

c. If ion temperature gradient is very steep (strong), and density profile is flat, what terms in the above equations can be ignored for a derivation of strong ITG instability? Identify as many non-essential terms as you can.

d. **(10pts)** Find a dispersion relation for ITG from a simplified equation in "c".

e. Compare the relative amplitude of fluctuations, $\delta T_i/T_i$ and $\delta n/n$.
Do density fluctuations exist for a flat mean density profile?
State the reason for your answer.