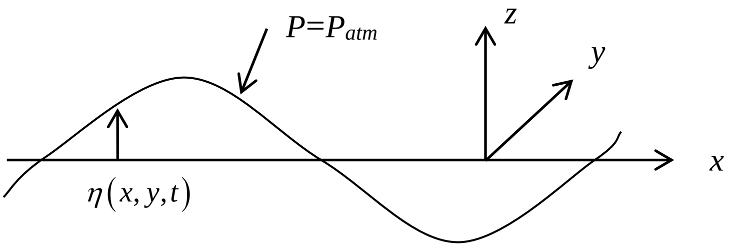
Final Exam Solution

1. (20%)

*** 1번문제 전체 기준:** 식 전개 과정에서 오류는 0.5 points 감점

(1) (10%) Nonlinear Free Surface Boundary Condition



Kinematic F.S.B.C.
If a geometric surface is written to
F(x,y,z,t) = 0, (e.g. $x^2+y^2+z^2-a^2=0$)

On the moving surface,
$\frac{dF}{dt}=0$ all the time $\Rightarrow$ $\frac{\partial F}{\partial t}+\vec{u}\cdot\nabla F=0$ (where $\vec{u}$ is moving speed)

On free surface,
$F=z-\eta=0$,

$$\Rightarrow \frac{DF}{Dt}=\frac{d}{dt}(z-\eta)=0$$

$$\Rightarrow \frac{\partial}{\partial t}(z-\eta)+\vec{u}\cdot\nabla(z-\eta)=0$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0,\ \frac{\partial \eta}{\partial z}=0\ \left(\because \eta=f(x,y,t),\frac{\partial z}{\partial z}=1\right)$$

Thus,

$$\boxed{\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y}-\frac{\partial \phi}{\partial z}=0}\ on\ z=\eta$$

<**채점기준 (Nonlinear K.F.S.B.C.: Total 5 points)**>

- $\frac{\partial \eta}{\partial z}=0$인 이유 포함하여 유도 완료한 경우 5 points
- 미분과정에서 $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0,\frac{\partial \eta}{\partial z}=0$를 명시하지 않은 경우 4 points
- Nonlinear 유도하지 않고 결과만 Linear에 쓴 경우 1 points

Dynamic F.S.B.C.
Bernoulli's Equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{P_{atm}}{\rho} + gz = C(t)$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + gz = C(t) - \frac{P_{atm}}{\rho}$$

When $\phi = 0 \Rightarrow \eta = 0$, then $C(t) - \frac{P_{atm}}{\rho} = 0$

Thus,

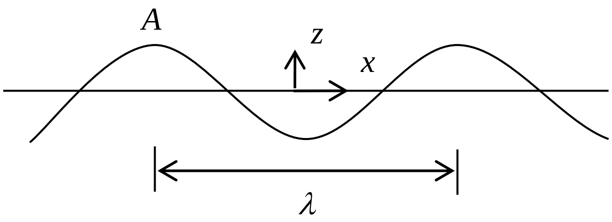
$$\boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + gz = 0} \; on \; z = \eta$$

<채점기준 (Nonlinear D.F.S.B.C.: Total 5 points)>

- Far field에서 $\phi = 0, \eta = 0$이라고 명시해야 — 5 points
- Far field만 언급하고 $C(t) - \frac{P_{atm}}{\rho} = 0$이라고 서술 — 4 points
- $\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + gz = C(t) - \frac{P_{atm}}{\rho}$이후로 별다른 설명 없이 결론을 낸 경우 — 3 points
- Nonlinear 유도하지 않고 결과만 Linear에 쓴 경우 — 1 point

(2) (10%) Linearized Free Surface Boundary Condition

Assumption: kA << 1 (small wave slope)



$\frac{A}{\lambda} = O(\varepsilon),\ \varepsilon << 1 \Rightarrow$ Small disturbance

Linearization: Taylor Series Expansion

$$f(x) = f(x_0) + (x - x_0)\left.\frac{df}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2 f}{dx^2}\right|_{x=x_0} + \cdots$$

Kinematic F.S.B.C.

If we assume that $\eta = O(\varepsilon),\ \phi = O(\varepsilon)$ and neglect terms more than second order, i.e. $\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} = O(\varepsilon^2)$

$$\left(\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\phi}{\partial z}\right)_{z=\eta} = \left(\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\phi}{\partial z}\right)_{z=0}$$

$$+(\eta - 0)\frac{\partial}{\partial z}\left(\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\phi}{\partial z}\right)_{z=0}$$

$$+\frac{(\eta - 0)^2}{2!}\frac{\partial^2}{\partial z^2}\left(\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\phi}{\partial z}\right)_{z=0} + \cdots = 0$$

$$\Rightarrow O(\varepsilon) : \boxed{\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} = 0} \quad \text{on } z = 0$$

<채점기준(Linear K.F.S.B.C.: Total 4 points)> (감점사유)

- kA<<1 명시한 경우 감점 없음
  - Small amplitude만 언급한 경우 — 0.5 point 감점
  - Small amplitude도 언급하지 않은 경우 — 1 point 감점
- Taylor 전개하지 않은 경우 — 2 points 감점
- Taylor 전개 진행과정에서, 어떤 항을 neglect 했는지 명시 않은 경우 — 0.5 point 감점
- High order term을 neglect 하는 과정에서, order를 전혀 명시 않은 경우 — 0.5 point 감점

Dynamic F.S.B.C.

If we assume that $\eta = O(\varepsilon),\ \phi = O(\varepsilon)$ and neglect terms more than second order, i.e. $\nabla\phi\cdot\nabla\phi = O(\varepsilon^2)$

$$\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+gz\right)_{z=\eta} = \left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+gz\right)_{z=0}$$
$$+(\eta-0)\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+gz\right)_{z=0}$$
$$+\frac{(\eta-0)^2}{2!}\frac{\partial^2}{\partial z^2}\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+gz\right)_{z=0}+\cdots=0$$

$$\Rightarrow O(\varepsilon):\boxed{\frac{\partial\phi}{\partial t}+g\eta=0} \quad \text{on } z=0$$

<채점기준(Linear D.F.S.B.C.: Total 4 points)> (감점사유)

- Taylor 전개하지 않은 경우 2 points 감점
- Taylor 전개 과정에서, 어떤 항 neglect 했는지 명시하지 않은 경우 0.5 point 감점
- High order term에서 order를 명시하지 않은 경우 0.5 point 감점
- gz를 $g\eta$로 Taylor 전개한 경우 1 point 감점
- $g\eta$를 첫 번째 Taylor항으로 구한 경우 0.5 point 감점

Combining two conditions,

$$\frac{\partial\phi}{\partial t}+g\eta=0 \Leftrightarrow \frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\eta}{\partial t}=0$$
$$\frac{\partial\eta}{\partial t}-\frac{\partial\phi}{\partial z}=0 \Leftrightarrow \frac{\partial\eta}{\partial t}=\frac{\partial\phi}{\partial z}$$
$$\boxed{\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=0} \quad \text{on } z=0$$

<채점기준(Combining: Total 2 points)>

- 미분을 하지 않은 채로 두 개의 식을 정리한 경우 0.5 point

2. (25%)

(1) (5%)

In deep water, $\omega^2 = gk$

$$\therefore \lambda = g\frac{2\pi}{\omega^2} = \frac{gT^2}{2\pi} = \boxed{156.13m}$$

$$\phi = \frac{gA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t) = \frac{gA}{\omega}e^{kz}\sin(kx-\omega t)\ \ (in\ deep\ water)$$

$$u = \frac{\partial\phi}{\partial x} = \frac{gA}{\omega}ke^{kz}\cos(kx-\omega t) = \omega Ae^{kz}\cos(kx-\omega t)$$

$$u_{\max} = \left.\frac{\partial\phi}{\partial x}\right|_{\max} = \omega Ae^{kz}$$

At $z = -20m$, $u_{\max} = 3m/s$

$$\therefore A = \frac{u_{\max}}{\omega e^{kz}} = \boxed{10.68m}$$

<채점기준>

- 최종 결과까지 적절히 구한 경우 5 points
- z=0으로 계산한 경우 3.5 points

(2) (5%)

Phase velocity: $V_p = \frac{\lambda}{T} = \boxed{15.6\,m/s}$

Group velocity: $V_g = \frac{1}{2}V_p = \boxed{7.8\,m/s}$ $(\because\ deep\ water)$

(3) (5%)

$$w = \frac{\partial\phi}{\partial z} = \frac{gA}{\omega}ke^{kz}\sin(kx-\omega t) = \omega Ae^{kz}\sin(kx-\omega t)$$

$$w_{\max} = \left.\frac{\partial\phi}{\partial z}\right|_{\max} = \omega Ae^{kz} = \boxed{3\text{ m/s}}$$

$$or\ \left(w_{\max} = u_{\max}\text{ in deep water } \Rightarrow\ w_{\max} = \omega Ae^{kz} = 3\text{ m/s}\right)$$

<채점기준> (부분점수)

- z 방향 속도를 적절히 구한 경우 + 2 points
- z=-20 대입하고, 계산과정 보임 + 1 points
- 최종 답안 + 2 points

(4) (5%)

$$P_{dynamic} = -\rho\frac{\partial\phi}{\partial t} = \rho g A\frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\omega t)$$

$$kh\to\infty; P_{dynamic} = \rho g A e^{kz}\cos(kx-\omega t)$$

$$\left|P_{dynamic}\right|_{z=-20m} = \left|\rho g A e^{kz}\right|_{z=-20m} \approx \boxed{48.019\ \text{kPa}\ \left(\text{kN/m}^2\right)}$$

<**채점기준**> **(부분점수)**

- Dynamic pressure 식을 적절히 구한 경우 + 2 points
- z=-20 대입하여 계산과정 보임 + 1 points
- 최종 답안 + 2 points

(5) (5%)

$$\overline{E} = \frac{1}{2}\rho g A^2\cdot 1 = \boxed{573.463\,\text{kJ/m}^2}$$

<**채점기준**> **(부분점수)**

- Energy density 식 적절히 쓴 경우 + 3 points
- 최종 답안 + 2 points

3. (35%)

(1) (5%)

Significant wave height: $\dfrac{4\times2100+5\times1400+6\times900+7\times500+8\times100}{5000}=\boxed{5.02m}$

1/10 highest wave amplitude (**<u>not</u>** averaged wave amplitude): $\dfrac{1}{2}\times 6m=\boxed{3m}$

<**채점기준**>

- Significant wave height (total: 3 points)

최종 결과까지 적절히 구한 경우	3 points
Amplitude와 height를 혼동한 경우	2 points

- 1/10 highest wave amplitude (total: 2 points)

최종 결과까지 적절히 구한 경우	2 points
Amplitude와 height를 혼동한 경우	1 points

(2) (10%)

$$H_{1/3}=4\sqrt{m_0}\Rightarrow m_0=\left(\frac{H_{1/3}}{4}\right)^2=1.575$$

$$A=\sqrt{2m_0\ln N}=\sqrt{2\times1.575\times\ln\left(\frac{365\times24\times3600\times100}{8}\right)}=7.896m$$

$$H_{design}=2A=\boxed{15.792m}$$

<**채점기준**>

- 최종 결과까지 적절하게 구한 경우	10 points
- 계산 오류(mo, H 등 잘못된 값으로 계산: 3점 감점)	7 points

(3) (15%)

$$\phi = \frac{gA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t) = \frac{gA}{\omega}e^{kz}\sin(kx-\omega t)\ \ (in\ deep\ water)$$

$$u = \frac{\partial \phi}{\partial x} = \frac{gA}{\omega}ke^{kz}\cos(kx-\omega t) = \omega Ae^{kz}\cos(kx-\omega t)$$

$$\frac{du}{dt} = \omega^2 Ae^{kz}\sin(kx-\omega t)$$

Using u and $\frac{du}{dt}$ at x=0. Assume that $C_M = 2, C_D = 1$.

Inertial force (for a truncated cylinder in deep water):

$$F_I = \int_{-l}^{0} dF_I = \int_{-l}^{0}\left\{\rho C_M \frac{\pi d^2}{4}\frac{du}{dt}\right\}dz$$

$$= -\int_{-l}^{0}\left\{\rho C_M \frac{\pi d^2}{4}A\omega^2 e^{kz}\sin\omega t\right\}dz = -\rho C_M \frac{\pi d^2}{4}A\omega^2 \frac{1}{k}\left(1-e^{-kl}\right)\sin\omega t\ \ (where\ l\ is\ draft\ of\ the\ spar)$$

$$|F_I| = \rho C_M \frac{\pi d^2}{4}A\omega^2\frac{1}{k}\left(1-e^{-kl}\right) = 1025\times 2.0\times\frac{\pi\times 20^2}{4}\times 7.896\times 0.524^2\times\frac{1}{0.028}\times\left(1-e^{-0.028\times 150}\right) = \boxed{4.913\times 10^7\,N}$$

Drag force (for a truncated cylinder in deep water):

$$F_D = \int_{-l}^{0}\left\{\frac{1}{2}\rho C_D du|u|\right\}dz$$

$$= \int_{-l}^{0}\left\{\frac{1}{2}\rho C_D d(A\omega)^2 e^{2kz}\cos\omega t|\cos\omega t|\right\}dz = \frac{1}{2}\rho C_D d(A\omega)^2\frac{1}{2k}\left(1-e^{-2kl}\right)\cos\omega t|\cos\omega t|$$

$$|F_D| = \frac{1}{2}\rho C_D d(A\omega)^2\frac{1}{2k}\left(1-e^{-2kl}\right) = \frac{1}{2}\times 1025\times 1.0\times 20\times(7.896\times 0.524)^2\times\frac{1}{2\times 0.028}\times\left(1-e^{-2\times 0.028\times 150}\right) = \boxed{3.134\times 10^6\,N}$$

<채점기준> (감점사유)

- **배점:** Maximum drag force / maximum inertia force 각각 7.5 points 부여

적분 오류	2 points 감점
계산 오류(답이 틀린 경우)	2 points 감점
inertia force 적분 시, 부호 오류	1 points 감점

(4) (5%)

Strouhal number: $S_t = 0.21\ \left(*Rn = \frac{Ud}{\nu} = \frac{2.0\times 20}{1.0\times 10^{-6}} = 4\times 10^7\right)$

$$f = \frac{S_t U}{d} = \frac{0.21\times 2}{20} = 0.021\ \text{Hz}$$

$$T = 1/f = \boxed{47.62\,\text{sec}}$$

<채점기준>

- 최종 결과까지 적절하게 구한 경우	5 points
- 세부 계산과정(Strouhal number, 속도 값 등) 부족	4 points
- frequency까지만 구한 경우	4 points
- 계산 오류	3 points

4. (20%)

(1) (7%)

$\omega^2 = gk \tanh kh$

1st mode: $k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi}{2\times37}$, $h_1 = 35\times0.5$, $\omega_1 = \sqrt{gk_1 \tanh k_1 h_1} = \boxed{0.867\, rad/\sec}$

2nd mode: $k_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{37}$, $h_2 = 35\times0.5$, $\omega_2 = \sqrt{gk_2 \tanh k_2 h_2} = \boxed{1.287\, rad/\sec}$

<**채점기준**>

- 최종 결과까지 적절하게 구한 경우 7 points
- 과정 일부만 맞을 경우(최종 답 틀림) 3.5 points

(2) (5%)

$\eta = A\cos kx \cos \omega t$

$u = \frac{\partial \phi}{\partial x} = A\omega \frac{\cosh k(z+h)}{\sinh kh} \sin kx \sin \omega t \rightarrow x_p = \int u dt = -A \frac{\cosh k(z+h)}{\sinh kh} \sin kx \cos \omega t + x_0$

$w = \frac{\partial \phi}{\partial z} = -A\omega \frac{\sinh k(z+h)}{\sinh kh} \cos kx \sin \omega t \rightarrow z_p = \int w dt = A \frac{\sinh k(z+h)}{\sinh kh} \cos kx \cos \omega t + z_0$

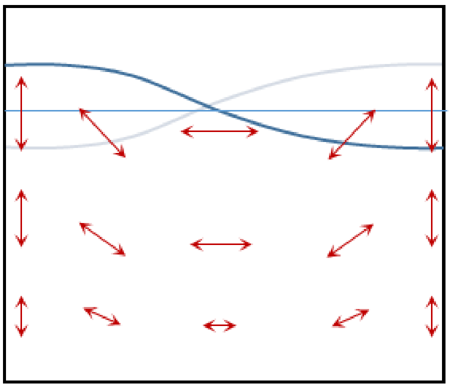
For example, (1st mode)

At $x = 0$ and $x = 37$, $x_p - x_0 = 0$: only vertical movement

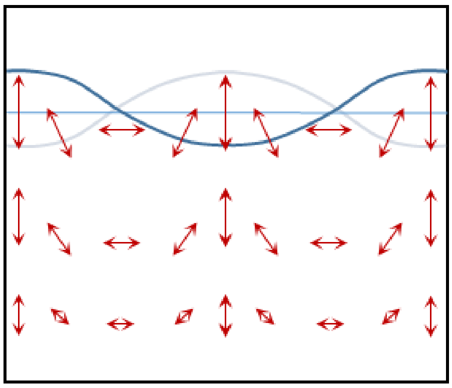
At $x = \frac{37}{2}$, $z_p - z_0 = 0$: only horizontal movement

At the other region, diagonal movement. Direction of the movement is determined by sin(kx) and cos(kx)

1st mode



2nd mode



<**채점기준**>

*** Sketch만 정확히 하면 점수 부여, 1st mode, 2nd mode 각각 3.5 points 부여**

- Free surface

    적절하게 sketch 한 경우 / 부족한 경우 각 1.0 / 0.5 points 차등부여

- particle의 방향벡터

    적절하게 sketch 한 경우 / 부족한 경우 각 1.5 / 1.0 / 0.5 points 차등부여

(3) (8%)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gk\tanh(kh)}} \Rightarrow h = \frac{1}{k}\tanh^{-1}\left\{\frac{1}{gk}\left(\frac{2\pi}{T}\right)^2\right\}$$

$$h_1 = \frac{1}{k_1}\tanh^{-1}\left\{\frac{1}{gk_1}\left(\frac{2\pi}{T}\right)^2\right\} = \frac{1}{\left(\frac{2\pi}{2\times 37}\right)}\tanh^{-1}\left\{\frac{1}{9.81\times\left(\frac{2\pi}{2\times 37}\right)}\left(\frac{2\pi}{10}\right)^2\right\} = \boxed{6.067m} \text{ : this filling depth should be avioded}$$

$$h_2 = \frac{1}{k_2}\tanh^{-1}\left\{\frac{1}{gk_2}\left(\frac{2\pi}{T}\right)^2\right\} = 1.423m\left(\text{non-practical filling depth}\right)$$

<채점기준>

- 2^{nd} mode까지 구한 경우 8 points
- 1^{st} mode에 대해서만 적절히 구한 경우 7 points