Quiz #2 Solution

1.

(1) (5%)

Central limit theorem: If a random variable x can be expressed as the sum of a large number of independent random variables x_i

$$x = x_1 + x_2 + \cdots + x_N\,,\ N:\ \text{large}$$

Then the probability density function $f(x)$ of x is Gaussian (normal) distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(x-\overline{x}\right)^2}{2\sigma^2}\right)$$

where $\overline{x},\sigma^2$ are mean and variation.

<채점기준>

- Central limit theorem을 포함하여 적절하게 설명한 경우 5 points
- 가정의 일부만 기술한 경우 1 points

(2) (5%)

Area of the wave spectrum is proportional to the wave energy (mean wave energy density). So the wave spectrum is called “energy spectrum.”

$$\sum_k S_\eta\left(\omega_k\right)\Delta\omega_k = \sum_k \frac{1}{2}A_k^2\,,\ \overline{E} = \frac{1}{2}\rho g A^2$$

<채점기준>

- 적절하게 설명한 경우 5 points
- wave spectrum의 면적과 wave energy와의 관계가 모호한 경우 4 points

(3) (5%)

Phase velocity: $V_p = \dfrac{\lambda}{T} = \dfrac{\omega}{k}$, Developed wave crest advances with the phase velocity.

Group velocity: $V_g = \left(\dfrac{1}{2} + \dfrac{kh}{\sinh 2kh}\right)V_p$, Wave energy propagates with the group velocity.

<채점기준> (부분점수)

- phase velocity, group velocity 식 각 +1 points
- phase velocity, group velocity 설명 각 +1.5 points

(4) (5%)

$$V_g = \left(\frac{1}{2} + \frac{kh}{\sinh 2kh} \right) V_p$$

$$Deep\ water\left(kh \to \infty\right): V_g \simeq \left(\frac{1}{2} + 0 \right) V_p = \frac{1}{2} V_p \left(\because \frac{kh}{\sinh 2kh} \to 0 \right)$$

$$Shallow\ water\left(kh \to 0\right): V_g = \left(\frac{1}{2} + \frac{1}{2} \right) V_p = V_p \left(\because \frac{kh}{\sinh 2kh} \to \frac{kh}{2kh} \to \frac{1}{2} \right)$$

<채점기준> (부분점수)

- Group velocity 식 +1 points
- kh 범위에 따른 근사값 +1 points
- 수심별 Group velocity 각 +1.5 points

2.
(1) (10%)

Linear free surface boundary conditions

K.F.S.B.C.: $\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0$ on $z = 0$

D.F.S.B.C.: $\frac{\partial \phi}{\partial t} + g\eta = 0$ on $z = 0$

Galilean transformation: $\left.\frac{\partial}{\partial t}\right|_{X,Y,Z} = \left.\frac{\partial}{\partial t}\right|_{x,y,z} - U\left.\frac{\partial}{\partial x}\right|_{x,y,z}$

Kinematic free surface boundary condition

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\eta - \frac{\partial \phi}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \eta}{\partial t} - U\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0$$

Using steady flow condition $\frac{\partial}{\partial t} = 0$,

$$\boxed{U\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = 0}$$

Dynamic free surface boundary condition

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\phi + g\eta = 0 \quad \Rightarrow \quad \frac{\partial \phi}{\partial t} - U\frac{\partial \phi}{\partial x} + g\eta = 0$$

Using steady flow condition $\frac{\partial}{\partial t} = 0$, $\boxed{-U\frac{\partial \phi}{\partial x} + g\eta = 0 \quad \text{on } z = 0}$

Combine the two equations and write into one equation.

L.K.F.S.B.C.: $U\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z} = 0$ on $z = 0$

L.D.F.S.B.C.: $-U\frac{\partial \phi}{\partial x} + g\eta = 0$ on $z = 0$

From L.D.F.S.B.C.,

$$\frac{\partial}{\partial x}\left(-U\frac{\partial \phi}{\partial x} + g\eta\right) = 0 \Rightarrow -U\frac{\partial^2 \phi}{\partial x^2} + g\frac{\partial \eta}{\partial x} = 0 \Rightarrow \frac{\partial \eta}{\partial x} = \frac{U}{g}\frac{\partial^2 \phi}{\partial x^2} \quad \text{on } z = 0$$

Substitute above relation into L.K.F.B.C., then

$$U\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \quad \Rightarrow \quad \frac{U^2}{g}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} = 0 \text{ on } z = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{g}{U^2}\frac{\partial \phi}{\partial z} = 0 \text{ on } z = 0$$

Using non-dimensional parameter $x' = \frac{x}{L}, z' = \frac{z}{L}, \phi' = \frac{\phi}{UL}$

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{gL}{U^2}\frac{\partial \phi'}{\partial z'} = 0 \Rightarrow \boxed{\frac{\partial^2 \phi'}{\partial x'^2} + \frac{1}{Fr^2}\frac{\partial \phi'}{\partial z'} = 0 \text{ on } z = 0 \text{ where } Fr = Froude\ Number}$$

$i)$ $\mathrm{U} \to 0\,(Fr \to 0)$; $\phi_z = 0$

$ii)$ $\mathrm{U} \to \infty\,(Fr \to \infty)$; $\phi_{xx} = 0 \Rightarrow \phi = 0\,(periodicy\ in\ x)$ $(\phi_z \neq 0)$

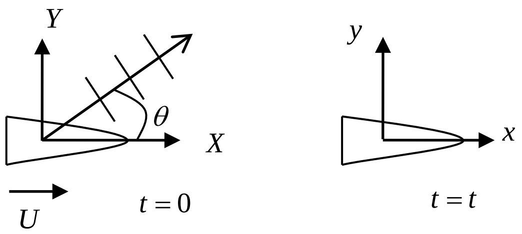
<채점기준> (감점요인)

Galilean transformation 부호 -1 points

U에 따른 결과 반대로 쓴 경우 -2 points

(2) (15%)

Ray Theory for Kelvin Wave



Wave propagation during *t* with angle θ

$(\text{distance during } t) = X\cos\theta + Y\sin\theta$

3D Plane progressive waves

$$\phi = \mathrm{Re}\left\{\frac{igA}{\omega}e^{(kz - ik(X\cos\theta + Y\sin\theta) + i\omega t)}\right\}$$

Substituting $X = x + Ut$

$$\phi = \mathrm{Re}\left\{\frac{igA}{\omega}e^{(kz - ik(x\cos\theta + y\sin\theta))}e^{-i(kU\cos\theta - \omega)t}\right\}$$

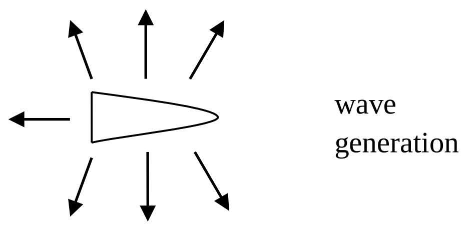
In steady flow,

$$\phi \neq \text{function of time, } t \Rightarrow kU\cos\theta - \omega = 0 \text{ or } \frac{\omega}{k} = V_p = \boxed{U\cos\theta} > V_g$$

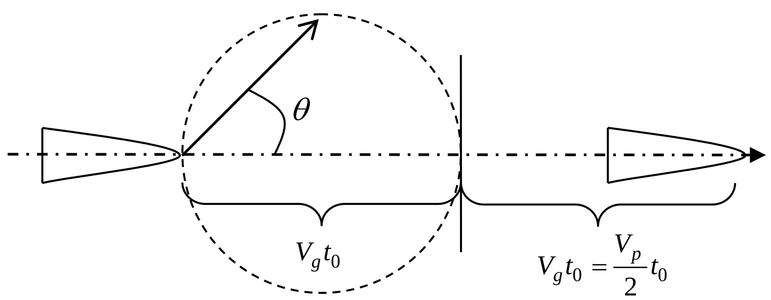
This is a phase velocity for steady waves in ship-fixed coordinate.

Ray Theory for Kelvin Wave

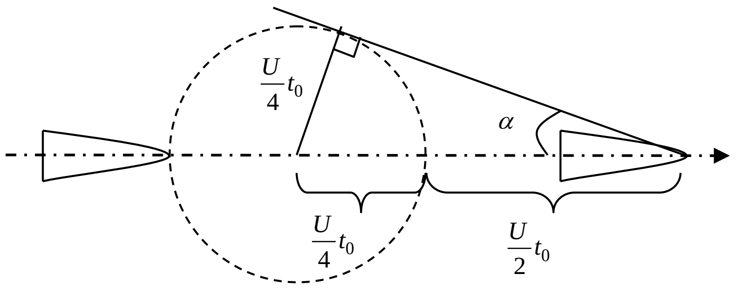
(i) at t=0



(ii) at t = t0 $(U = V_p = 2V_g)$



(iii) Kelvin Angle



$$\therefore \sin\alpha = \frac{\frac{1}{4}t_0}{\left(\frac{U}{2}+\frac{U}{4}\right)t_0} = \frac{1}{3} \Rightarrow \boxed{\alpha = \sin^{-1}\left(\frac{1}{3}\right)} = 19.47\,\text{deg} = 19°\,28'$$

<채점기준> (감점요인)

- Phase velocity for steady waves in ship-fixed coordinate $V_p = U\cos\theta$ 증명없이 사용 -2 points

- 유도 과정에서 $V_G = \frac{1}{2}V_P$ 언급하지 않은 경우 (설명 부족) -2 points

3.

(1) (10%)

$$\Delta\omega = \frac{1.35-0.35}{10} = 0.2;$$

i	ω_i	$S_\eta(\omega_i)$	$A_i = \sqrt{2S_\eta(\omega_i)\Delta\omega}$
1	0.45	$\frac{0.5+2.0}{2} = 1.25$	0.7071
2	0.65	$\frac{3.5+3.0}{2} = 3.25$	1.1402
3	0.85	$\frac{2.0+1.5}{2} = 1.75$	0.8367
4	1.05	$\frac{1.0+0.5}{2} = 0.75$	0.5477
5	1.25	$\frac{0.25+0.25}{2} = 0.25$	0.3162

<**채점기준**>

- 적절하게 모델링하고, 최종 답까지 잘 구한 경우 10 points
- $A_i = \sqrt{2S_\eta(\omega_i)\Delta\omega}$ 계산만 오류 8 points
- $\Delta\omega = 0.2$ 를 택하였으나, spectrum 계산이 정확하지 않은 경우 5 points
- $\Delta\omega = 0.1$을 택한 경우 (단순히 freqeuncy 만 선택한 경우) 2 points

(2) (10%)

$$\frac{1}{2}A^2 = \sum_i S_\eta(\omega_i)\Delta\omega_i = 0.5\times0.1 + 2.0\times0.1 + 3.5\times0.1 + \cdots + 0.25\times0.1 = 1.45$$

$$\boxed{A = \sqrt{2\cdot\sum_i S_\eta(\omega_i)\Delta\omega_i} = \sqrt{2\cdot1.45} = 1.7029m}$$

<**채점기준**>

- Spectrum 면적(Energy)이 보존되도록 모델링 하여 wave amplitude까지 잘 구한 경우 10 points
- Spectrum 면적에 대한 개념은 있으나, $H_{1/3}$등을 택한 경우 5 points

(3) (10%)

$$N = \frac{3.1536\times10^7\times M_{year}}{T_{mean}} = \frac{3.1536\times10^7\times50}{5.68}$$

$$m_0 = \sum_k S_\eta(\omega_k)\Delta\omega_k = 1.45$$

$$\boxed{A_{50-year} = \sqrt{2m_0\ln N} = 7.5087m}$$

<**채점기준**>

- 최종 답안까지 적절하게 구한 경우 10 points
- 개념은 알고 있으나, 계산 오류 7 points

4. (25%)

i	$H_{1/3}$ \ T_{mean}	3.0 sec	5.0 sec	7.0 sec	sum	$\frac{N_{H,i}}{N_{total}}$
1	**2.0 m**	150	200	140	490	0.49
2	**5.0 m**	50	400	50	500	0.50
3	**7.0 m**	0	10	0	10	0.01
	sum	200	610	190	1000	1
	$\frac{N_{T,i}}{N_{total}}$	0.20	0.61	0.19	1	

$$\overline{T}_{mean} = \sum_i T_{mean,i} \times \frac{N_{T,i}}{N_{total}} = 3.0\times 0.20 + 5.0\times 0.61 + 7.0\times 0.19 = 4.98\sec$$

$$Q(H_0) = \sum_i P(H_{1/3,i}) \exp\left(-2\left(\frac{H_0}{H_{1/3,i}}\right)^2 \right)$$

$$(R.H.S.) = \frac{\overline{T}_{mean}}{365\times 24\times 3600\times M_{year}} = \frac{4.98}{3.1536\times 10^7 \times 100} = 1.57915\times 10^{-9}$$

$$(L.H.S.) = \sum_i \frac{N_{H,i}}{N_{total}} \exp\left(-2\left(\frac{H_0}{H_{1/3,i}}\right)^2 \right) = 0.49\times\exp\left(-2\left(\frac{H_0}{2.0}\right)^2 \right) + 0.50\times\exp\left(-2\left(\frac{H_0}{5.0}\right)^2 \right) + 0.01\times\exp\left(-2\left(\frac{H_0}{7.0}\right)^2 \right)$$

$$\boxed{\therefore H_0 \simeq 19.59m}$$