Plasma Electrodynamics 1 June 16, 2009

Solutions of Final Exam

1. 1) O

   2) X (caused by ..... conducting fluid --> by a current I in a wire)

   3) X (surface --> volume, $J_{ms} \rightarrow J_{mv}$ or $J_{ms} = M \times \hat{n}$)

   4) O 5) X (is --> is not) 6) O 7) O

   8) X (Hysteresis --> Ohmic
   or produced by eddy currents --> caused by domain-wall motion and domain rotation)

   9) X (electric --> magnetic) 10) O

2. 1) Differential form: $\nabla \times H = J + \frac{\partial D}{\partial t}$

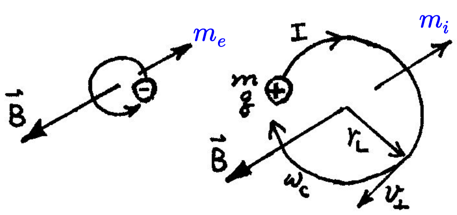
   Integral form: $\oint_C H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t} \right) \cdot ds = I_C + I_D$

   Meaning: The circulation of magnetic field intensity around any closed path is equal to the free and displacement currents flowing through the surface bounded by the path.

   2) The magnetic flux linking a surface S bounded by a contour C is

   $$\Phi = \int_S B \cdot ds = \int_S (\nabla \times A) \cdot ds = \oint_C A \cdot dl \quad \text{(Wb)}$$

   3) Magnetic moment $\boldsymbol{m}$ = (ring) current × area



   $= \text{IS} = |q| \frac{\omega_c}{2\pi} \left(\pi r_L^2 \right) = \frac{1}{2} |q| r_L v_\perp$

   Magnitude of $\boldsymbol{m}$: $m = \frac{1}{2} |q| r_L v_\perp = \frac{\frac{1}{2} m v_\perp^2}{B}$

   Direction of $\boldsymbol{m}$ is independent of q and antiparallel to $\boldsymbol{B}$ *(Diamagnetic)*.

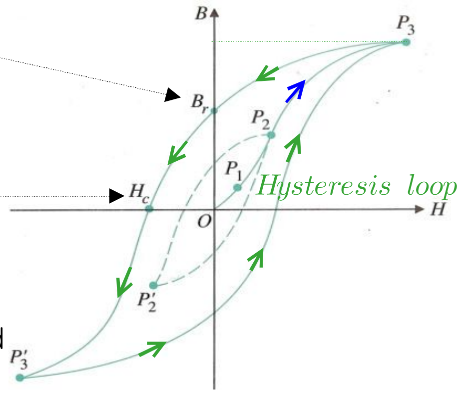
   $$\boldsymbol{m} = (\tfrac{1}{2})\, q \boldsymbol{r}_L \times \boldsymbol{v}_\perp = -\frac{(\frac{1}{2}) m v_\perp^2}{B} \frac{\boldsymbol{B}}{B}$$

4) Remnant flux density B_r:

The residual magnetic flux density that does not go to zero after the applied magnetic field is removed.

Coercive field intensity H_c:

The magnetic field intensity applied in the opposite direction to make the magnetic flux density of a magnetized medium vanish.



5) In a uniform magnetic field, $\boldsymbol{F}_m = I\oint d\boldsymbol{l}\times\boldsymbol{B} = 0$

$$\boldsymbol{T} = \boldsymbol{m}\times\boldsymbol{B} = \pi a^2 I\hat{\imath}\times\boldsymbol{B}$$

6) From $\nabla\cdot\boldsymbol{B}=0$, vector magnetic potential: $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ ①

① in Faraday's law $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$:

$$\nabla\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right)=0 \quad\Rightarrow\quad \boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t} = -\nabla V$$

$$\Rightarrow\quad \boldsymbol{E} = -\nabla V - \frac{\partial\boldsymbol{A}}{\partial t} \equiv \boldsymbol{E}_V + \boldsymbol{E}_A \quad \text{(V/m)} \qquad ②$$

② in Gauss's law $\nabla\cdot\boldsymbol{D} = \rho_v$:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} - \frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) \qquad ③$$

①, ② in Amprere's law $\nabla\times\boldsymbol{H} = \boldsymbol{J}+\dfrac{\partial\boldsymbol{D}}{\partial t}$ using $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$:

$$\nabla^2\boldsymbol{A} - \mu\varepsilon\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J} + \nabla\left(\nabla\cdot\boldsymbol{A} + \mu\varepsilon\frac{\partial V}{\partial t}\right) \qquad ④$$

Using Lorentz condition, $\nabla\cdot\boldsymbol{A} + \mu\varepsilon\dfrac{\partial V}{\partial t} = 0$ by Lorentz gauge transformations, ③ and ④ are simplified as

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} V \\ \boldsymbol{A}\end{Bmatrix} = \begin{Bmatrix} -\rho_v/\varepsilon \\ -\mu\boldsymbol{J}\end{Bmatrix}$$: Wave equation

3. 1) Axially symmetric $(\partial/\partial\phi = 0)$ and No edge effect $(\partial/\partial z = 0)$

$$\oint_C \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_o I \quad\Rightarrow\quad B(r)2\pi r = \mu_o I \quad\Rightarrow\quad \boldsymbol{B}(r) = \hat{\phi}\frac{\mu_o I}{2\pi r}$$

2) $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{d}{dr}\left(r\frac{\partial}{dr}\right)$$

$\nabla^2\boldsymbol{A} = -\mu_o\boldsymbol{J}$: vector Poisson's equation

BVP in a current-free $(\boldsymbol{J}=0)$ region $(a<r<b)$ $\Rightarrow$ $\nabla^2\boldsymbol{A} = 0$

$$\frac{d}{dr}\left(r\frac{dA_z}{dr}\right) = 0 \qquad ①$$

BCs:

$A_z(r)|_{r=b} = 0$ ②

$$\oint_C \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_o I \Rightarrow \oint_C (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{l} = \mu_o I \Rightarrow \oint\left(-\frac{\partial A_z}{\partial r}\hat{\phi}\right)\cdot(\hat{\phi}\, r\, d\phi) = \mu_o I$$

$$\Rightarrow \quad -\int_0^{2\pi}\left(\frac{\partial A_z}{\partial r}\right)(r\,d\phi) = \mu_o I \quad ③$$

Integrating ① twice, $r\dfrac{dA_z}{dr} = C_1 \Rightarrow dA_z = C_1\dfrac{dr}{r} \Rightarrow A_z(r) = C_1 \ln r + C_2$ ④

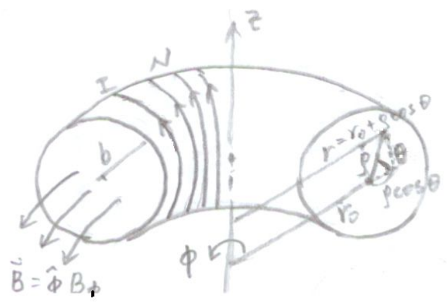
② in ④: $C_2 = C_1 \ln b$ ⑤, ⑤ in ④: $A_z(r) = C_1 \ln\dfrac{r}{b}$ ⑥

⑥ in ③: $-\displaystyle\int_0^{2\pi}\left(\frac{C_1}{r}\right)(r\,d\phi) = \mu_o I \Rightarrow -2\pi C_1 = \mu_o I \Rightarrow C_1 = -\mu_o I/2\pi$ ⑦

⑦ in ⑥: $A_z(r) = \hat{z}\dfrac{\mu_o I}{2\pi}\ln\dfrac{b}{r}$

$$\therefore \quad \boldsymbol{B}(r) = \nabla\times\boldsymbol{A} = -\hat{\phi}\frac{dA_z}{dr} = \hat{\phi}\frac{\mu_o I}{2\pi}\left(\frac{1}{r}\right), \quad (a < r < b)$$

4. 1) Ampere's circuital law: $\oint_C \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_o N I, \quad (b-a) < r < (b+a)$



$$\Rightarrow \boldsymbol{B}(r) = \hat{\phi}\frac{\mu_o N I}{2\pi}\frac{1}{r}, \quad (b-a) < r < (b+a)$$

2) $\Phi = \displaystyle\int \boldsymbol{B}\cdot d\boldsymbol{s} = \int_0^b\int_0^{2\pi} B_\phi\, \rho\, d\theta\, d\rho$

$$= \frac{\mu_o N I}{2\pi}\left[\int_0^b\left(\int_0^{2\pi}\frac{d\theta}{r_o + \rho\cos\theta}\right)\rho\, d\rho\right]$$

$$= \frac{\mu_o N I}{2\pi}\int_0^b \frac{2\pi}{\sqrt{r_o^2+\rho^2}}\rho\, d\rho$$

$$= -\mu_o N I\sqrt{r_o^2-\rho^2}\Big|_o^b = \mu_o N I\left(r_o - \sqrt{r_o^2-b^2}\right)$$

3) $\therefore \quad L = \dfrac{N\Phi}{I} = \mu_o N^2\left(r_o - \sqrt{r_o^2-b^2}\right)$

5. 1) Using Ampere's circuital law, $\boldsymbol{B}_1(y) = \hat{x}\dfrac{\mu_o I}{2\pi y}$

2) motional emf = $\oint(\boldsymbol{u}\times\boldsymbol{B}_1)\cdot d\boldsymbol{l}$

$$= \frac{\mu_o I_o u_o}{2\pi}\left[\int_\downarrow\left(\hat{y}\times\frac{\hat{x}}{d}\right)\cdot(-\hat{z}\,dl) + \int_\uparrow\left(\hat{y}\times\frac{\hat{x}}{d+w}\right)\cdot(\hat{z}\,dl)\right]$$

$$= \frac{\mu_o I_o u_o h}{2\pi}\left(\frac{1}{d}-\frac{1}{d+w}\right) = \frac{\mu_o I_o u_o h w}{2\pi d(d+w)}$$

3) $i_2 = -\dfrac{emf}{R} = -\dfrac{\mu_o I_o u_o h w}{2\pi d(d+w) R}$ flowing in the counter-clock-wise direction

to increase Φ in loop to oppose the decrease in Φ due to loop movement.