Name: Solutions

Computer Modeling (Spring 2008)

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Time: 75 minutes

Instructions:

1. This is a closed book examination. Calculators are allowed. Any form of cheating on the examination will result in a zero grade.
2. Do all your work inside this booklet, using the backs of pages if needed. The problems are of varying degrees of difficulty so please pace yourself and answer the questions in the order which best suits you. Answers to questions should be as brief and specific as possible.
3. Partial credit will be given even if your final solution is incorrect, if you show the intermediate steps you take in getting to the final solution. Clearly state any assumptions you make in your answers, and justify your assumptions.
4. Please write your answers neatly and, in case of numerical problems, *put the final answer in a box.* Good luck!

Problem No.	Max Points	Received Points
1	5	
2	5	
3	5	
4	10	
5	10	
6	10	
Total	45	

1. [**5 points**] When modeling and evaluating computer systems, what are the pros and cons of analytic methods? How about simulation-based methods?

2. [**5 points**] Prove the Markov property of the exponential distribution.

3. [**5 points**] Prove the Poisson distribution's equivalence with the exponential distribution.

4. **[10 points]** Consider a computing server with one processor where jobs arrive with the average rate of 2 per sec following the Poisson distribution. The time for serving a job follows the exponential distribution with the average of 0.4 sec. The system's queue is the infinite non-preemptive FIFO.

(a) What is the average number of jobs in the system at any time instant?

$$\begin{aligned} \rho &= 0.8 \\ L &= \frac{\rho}{1-\rho} = 4 \end{aligned}$$

(b) How long does a job spend in the system on average?

$$W = \frac{L}{\lambda} = 2sec$$

(c) What is the average length of the queue?

$$\begin{aligned} W_q &= W - W_s = 2 - 0.4 = 1.6sec \\ L_q &= \lambda W_q = 3.2 \end{aligned}$$

(d) What is the probability that at least two jobs are in the queue?

$$Prob(Q \geq 2) = Prob(N \geq 3) = P_3 + P_4 + \cdots = \rho^3 = 0.512$$

5. **[10 points]** Consider a computer system with TWO processors and NO waiting queue. Out of the two processors, one is faster than the other. That is, the average execution time of a job on the fast processor is 0.2 sec while that on the slow processor is 0.5 sec. Computational jobs arrive at this system with the average rate of 5 jobs/sec following the Poisson distribution. If both processors are idle, a job is assigned to the fast processor. Only when the fast processor is busy, a job is assigned to a slow processor. Once a job is assigned to a processor, it must execute on that processor until completion.

   (a) What is the average number of jobs in the system at any time instant?

   $$0P_1 + 1P_2 + 1P_3 + 2P_4 = 1.0082$$

   (b) What is the probability that the fast processor is busy?

   $$P_2 + P_4 = 0.499$$

   (c) What is the probability that the slow processor is busy?

   $$P_3 + P_4 = 0.5083$$

   (d) What is the average time for completing a job?

   $$W = L/\lambda_a = 1.0082/(5(1 - P_4)) = 0.286sec$$

   (Hint: Draw system's state transition diagram. In a statistically equilibrium, from the perspective of a state, the total outgoing rate is equal to the total incoming rate.)

   Solution: Let

   - State 1: (0,0) which means both the fast and slow processors are idle
   - State 2: (1,0) which means only the fast processor is busy
   - State 3: (0,1) which means only the slow processor is busy
   - State 4: (1,1) which means both the fast and slow processors are busy.

   In statistically equilibrium,

   $$\begin{aligned} 5P_1 &= 5P_2 + 2P_3 \\ 5P_2 + 5P_2 &= 5P_1 + 2P_4 \\ 5P_3 + 2P_3 &= 5P_4 \\ 5P_4 + 2P_4 &= 5P_2 + 5P_3 \end{aligned}$$

   Also,

   $$P_1 + P_2 + P_3 + P_4 = 1$$

Solving the above equalities,

$$\begin{aligned} P_1 &= 0.2882 \\ P_2 &= 0.2034 \\ P_3 &= 0.2118 \\ P_4 &= 0.2965 \end{aligned}$$

From this, we can answer the above sub-questions.

6. **[10 points]** Consider a queueing network with four computing facilities, S_1, S_2, S_3, and S_4. Each computing facility has an infinite non-preemptive FIFO queue and one processor. Jobs arrive at S_1 following the Poisson distribution with the average rate of 2 jobs/sec. Once a job completes at S_1, it either leaves the system or moves to S_2, with the probabilities of 0.7 and 0.3, respectively. Once completed at S_2, it moves to either S_3 or S_4 with the probabilities of 0.8 and 0.2, respectively. Once a job completes at either S_3 or S_4, it reenters S_1. The execution times of a job by the processors of S_1, S_2, S_3, and S_4 follow the exponential distribution with the averages of 0.2, 0.1, 0.1, and 0.25, respectively.

   (a) What are the average numbers of jobs in S_1, S_2, S_3, and S_4?

   (b) What are the average times a job spend in S_1, S_2, S_3, and S_4 per visit?

   (c) What is the average number of jobs in the system?

   (d) How long does a job spend in the system on average?

   Solution:

$$
\begin{aligned}
\Lambda_1 &= \lambda + \Lambda_3 + \Lambda_4 \\
\Lambda_2 &= 0.3\Lambda_1 \\
\Lambda_3 &= 0.8\Lambda_2 \\
\Lambda_4 &= 0.2\Lambda_2
\end{aligned}
$$

Solving the above equalities,

$$
\begin{aligned}
\Lambda_1 &= 2.68, \rho_1 = 0.572 \\
\Lambda_2 &= 0.858, \rho_2 = 0.0858 \\
\Lambda_3 &= 0.686, \rho_3 = 0.0686 \\
\Lambda_4 &= 0.172, \rho_4 = 0.0429
\end{aligned}
$$

$$
\begin{aligned}
L_1 &= \frac{\rho_1}{1-\rho_1} = 1.34 \\
L_2 &= \frac{\rho_2}{1-\rho_2} = 0.094 \\
L_3 &= \frac{\rho_3}{1-\rho_3} = 0.074 \\
L_4 &= \frac{\rho_4}{1-\rho_4} = 0.045
\end{aligned}
$$

$$
\begin{aligned}
W_1 &= \frac{L_1}{\Lambda_1} = 0.47 \\
W_2 &= \frac{L_2}{\Lambda_2} = 0.1096 \\
W_3 &= \frac{L_3}{\Lambda_3} = 0.1079 \\
W_4 &= \frac{L_4}{\Lambda_4} = 0.2622
\end{aligned}
$$

$$L = L_1 + L_2 + L_3 + L_4 = 1.553$$

$$W = L/\lambda = 0.7765$$