Midterm Examination
Subject: Optimal Design
Date: 2008. 04. 17

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1. Consider a matrix $\underset{\sim}{\mathbf{A}} = \left[a_{ij} \right] = \underset{\sim}{\mathbf{A}}^{\mathbf{T}}$.

   A. Write down the definition of Positive-Definiteness for $\underset{\sim}{\mathbf{A}}$ [5pts.]

   B. Check if $\underset{\sim}{\mathbf{A}} = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$ is Positive-Definite using eigenvalues of $\underset{\sim}{\mathbf{A}}$ [5pts.]

   C. Check if $\underset{\sim}{\mathbf{A}} = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 6 \end{bmatrix}$ is Positive-Definite by Sylvester's test. [5pts.]

2. Prove if $f \in C^2$ and $f =$ convex $\leftrightarrow f''(x) \geq 0$ for $x \in S$ (S: convex set) [10pts.]

3. It is known that in the Golden Section Method, there exists a relation for interval I_n such that $I_{n+1} = \tau I_n$.

   A. Derive the value τ . [10pts.]

   B. What is the advantage using the Golden Section Method compared with the interval halving method? [10pts.]

4. In the Conjugate Gradient Method, the variable x is updated in the conjugate direction, $\mathbf{d}^{\mathbf{k}+1}$. In order to obtain $\mathbf{d}^{\mathbf{k}+1}$, $\beta^{\mathbf{k}}$ is derived as:

$$\beta^{\mathbf{k}} = \frac{\left(\mathbf{g}^{\mathbf{k}+1}\right)^{\mathbf{T}} \underset{\sim}{\mathbf{A}} \left(\mathbf{d}^{\mathbf{k}}\right)}{\left(\mathbf{d}^{\mathbf{k}}\right)^{\mathbf{T}} \underset{\sim}{\mathbf{A}} \left(\mathbf{d}^{\mathbf{k}}\right)} \qquad \text{(a)}$$

   However, when expression (a) is extended for Non-quadratic functions, $\beta^{\mathbf{k}}$ should be replaced by:

$$\beta^{\mathbf{k}} = \frac{\left(\mathbf{g}^{\mathbf{k}+1}\right)^{\mathbf{T}} \mathbf{g}^{\mathbf{k}}}{\left(\mathbf{g}^{\mathbf{k}}\right)^{\mathbf{T}} \mathbf{g}^{\mathbf{k}}} \qquad \text{(b)}$$

When modifying (a) into (b), equations (c)-(e) below will be used:

$$\left(\mathbf{g}^{\mathbf{k}+1}\right)^{\mathbf{T}} \mathbf{d}^{\mathbf{k}} = 0 \qquad \text{(c)}$$

$$\left(\mathbf{d}^{\mathbf{k}}\right)^{\mathbf{T}} \mathbf{g}^{\mathbf{k}} = -\left(\mathbf{g}^{\mathbf{k}}\right)^{\mathbf{T}} \mathbf{g}^{\mathbf{k}} \qquad \text{(d)}$$

$$\left(\mathbf{g}^{\mathbf{k}+1}\right)^{\mathbf{T}} \mathbf{g}^{\mathbf{k}} = 0 \qquad \text{(e)}$$

A. Prove equation (d). [10pts.]

B. Using equations (a), (c)-(e), derive equation (b). (equations (c) and (e) need not be proven.) [10pts.]

5. Consider the following problem:

$$\min f(x_1, x_2) = \frac{1}{4}x_1^2 - 2x_2 + x_2^2 + 1 \qquad \text{(f)}$$

We wish to solve (f) with $\mathbf{x}_o = [4, 0]$ using the Steepest Descent Method (SDM) and the Conjugate Gradient Method (CGM).

A. Perform 2 iterations using the Steepest Descent Method (SDM). [10pts.]
(You can use any method for the 1-D search, graphical or analytical ones.)

B. Perform 2 iterations using Conjugate Gradient Method (CGM). [10pts.]
(You can use any method for the 1-D search, graphical or analytical ones.)

C. Solve the minimization problem (f) again analytically. Show that the solution is a true minimum. [5pts.]

D. Compare the results of A-C. If there exists any difference, explain why.

E. Is there any way to improve the convergence of SDM in this problem? If so, explain it and re-write problem (f). (You do not have to perform any numerical iteration.) [5pts.]